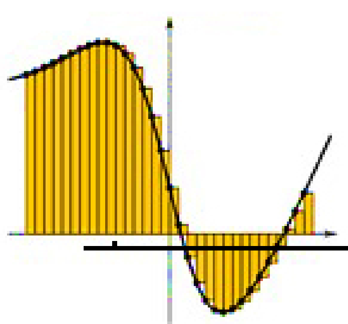
محاسبات عددی

(حل عددی معادلات)

عباس روحانی

استادیار دانشکده کشاورزی گروه مهندسی مکانیک بیو سیستم



حل عددی معادلات $f(x)=0$

هر گاه $f(\alpha)=0$ باشد آنگاه α را یک ریشه معادله می نامیم و یا می گوییم α یک صفر تابع f است.

حل تحلیلی معادلاتی مانند معادله درجه دوم $ax^2+bx+c=0$ به روشهای کلاسیک قابل حل است.

$$ax^2+bx+c=0=a(x-r_1)(x-r_2) \qquad r_1, r_2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$ax^2 + bx + c = 0$$

Dividing both sides by a, $(a \neq 0)$, we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Note if $a = 0$, the solution to

$$ax^2 + bx + c = 0$$

is

$$x = -\frac{c}{b}$$

Rewrite

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

as

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

یادآوری

$$x^3 + ax^2 + bx + c = 0 = (x - r_1)(x - r_2)(x - r_3)$$

First, calculate,

$$Q = \frac{3b - a^2}{9} \qquad R = \frac{9ab - 27c - 2a^3}{54} \qquad S = \sqrt[3]{R + \sqrt{Q^3 + R^2}} \qquad T = \sqrt[3]{R - \sqrt{Q^3 + R^2}}$$

Then the roots,

$$r_1 = S + T - \frac{a}{3} \qquad r_2 = \frac{S+T}{2} - \frac{a}{3} + \frac{j\sqrt{3}}{2}(S - T) \qquad r_3 = \frac{S+T}{2} - \frac{a}{3} - \frac{j\sqrt{3}}{2}(S - T)$$

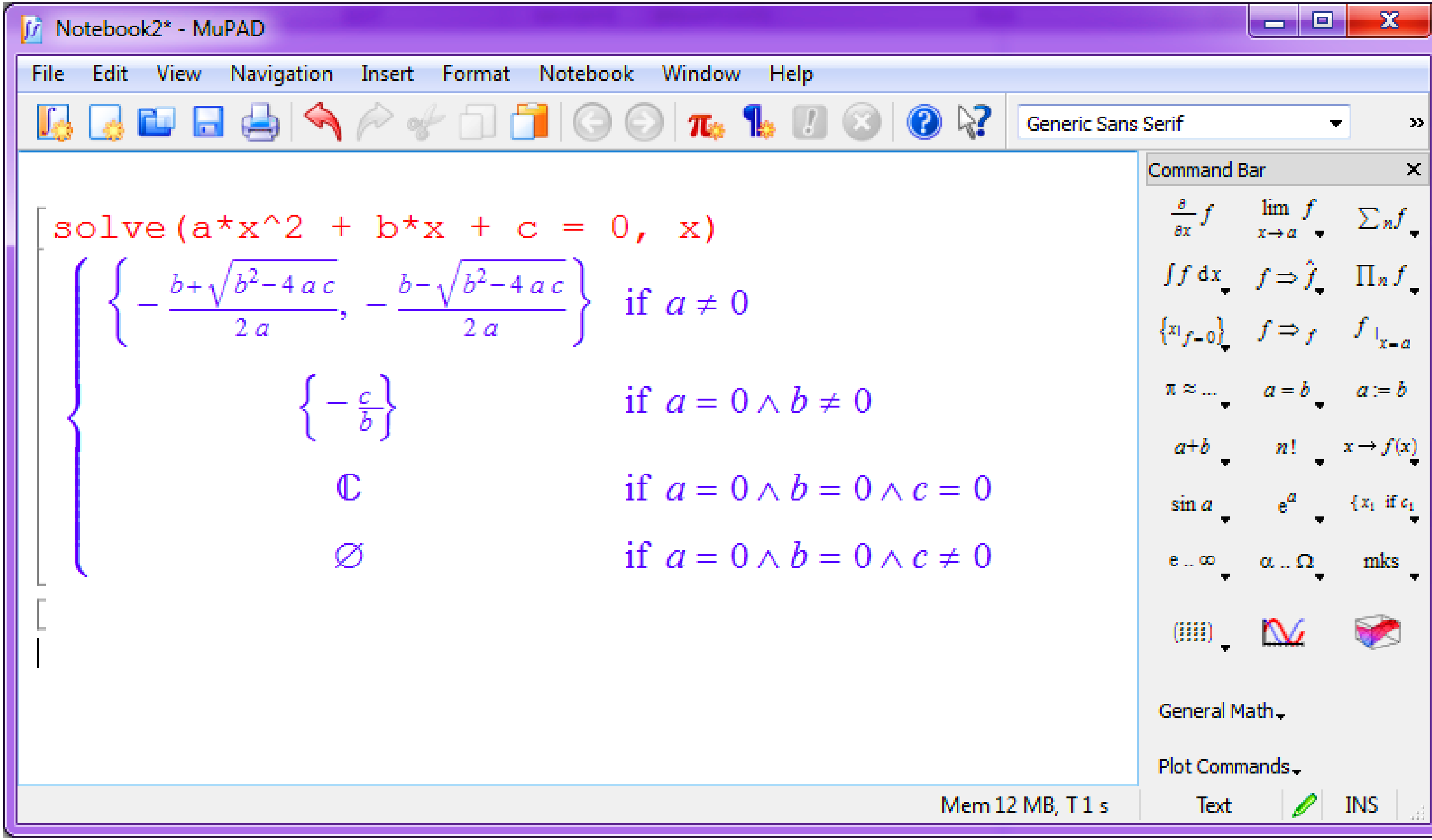
```
>> syms a  b  c  x
>> solve( a * x^2 + b * x + c == 0 , x )

ans =
        -( b + ( b^2 – 4 * a * c ) ^ (1/2) ) / ( 2*a )

         -( b - ( b^2 – 4 * a * c ) ^ ( 1/2 ) ) / ( 2*a )



>> solve( x^3 + a * x^2 + b * x + c == 0 , x )
```


Notebook2* - MuPAD
File Edit View Navigation Insert Format Notebook Window Help
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solve(a*x^2 + b*x + c = 0, x)
$\left\{-\frac{b+\sqrt{b^2-4ac}}{2a}, -\frac{b-\sqrt{b^2-4ac}}{2a}\right\}$ if $a \neq 0$
$\left\{-\frac{c}{b}\right\}$ if $a = 0 \wedge b \neq 0$
$\mathbb{C}$ if $a = 0 \wedge b = 0 \wedge c = 0$
$\emptyset$ if $a = 0 \wedge b = 0 \wedge c \neq 0$
General Math
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```
solve(a*x^2 + b*x + c = 0, x) assuming a > 0
```

$$\left\{-\frac{b+\sqrt{b^2-4\,a\,c}}{2\,a},\ -\frac{b-\sqrt{b^2-4\,a\,c}}{2\,a}\right\}$$

```
solve([x^2 + x*y + y^2 = 1, x^2 - y^2 = 0], [x, y])
```

$$\left\{\left[x=-\frac{\sqrt{3}}{3},\ y=-\frac{\sqrt{3}}{3}\right],\ \left[x=\frac{\sqrt{3}}{3},\ y=\frac{\sqrt{3}}{3}\right],\ [x=-1,\ y=1],\ [x=1,\ y=-1]\right\}$$

```
solve(x^4 - a = 0, x) assuming a = 16 and x in R_
```

$$\{-2,\ 2\}$$

؟ معادلاتی مانند زیر، قابل حل با روشهای تحلیلی نیستند و برای آنها باید از روشهای تقریبی استفاده کرد:

$$e^{-x} - cosx = 0$$
$$x + cosx = 0$$
$$x^2 - (1-x)^5 = 0$$

معمولا برای تعیین ریشه از یک معادله با دقت مورد نطر، لازم است تقریبی از آن ریشه یا فاصله کوچکی را که حاوی آن ریشه باشد، معلوم کرد.

بنابراین محدودیتهای زیر را در نظر می گیریم:

محدودیت الف:

فاصله ای موجود باشد که شامل ریشه باشد.

- تابع $y=f(x)$ در فاصله [a b] پیوسته باشد.
- $f(a)$ و $f(b)$ مختلف العلامت باشند، یعنی $f(a)f(b)<0$.

با وجود این دو شرط و براساس قضیه مقدار میانی،
عددی مانند α در فاصله [a b] وجود دارد، به طوری که $f(\alpha)=0$

محدودیت ب:

باید ریشه در فاصله مورد نظر یکتا باشد.
برای هر $x\in[a\ b]$:

$$f'(x) \neq 0$$

تعیین ریشه ها با دقت مورد نظر

با مشخص بودن فاصله ای که شامل یک ریشه معادله $f(x)=0$ است، برای تعیین تقریبی از ریشه مورد نظر با دقت مطلوب، دنباله ای از اعداد مانند x_n می سازیم بطوریکه با افزایش n، مقدار x_n به α نزدیک شود، یعنی

$$\lim_{n\to\infty} x_n = \alpha$$

با توجه به تعریف حد، عددی مانند N وجود دارد که

$$x_N \cong \alpha$$

معیار توقف محاسبه x_nها:

تعیین Nای که تقریب لازم را بدهد

معیارهای توقف الگوریتم

الف– هرگاه ϵ عددی معلوم باشد(مثلا $\epsilon=10^{-6}$)، x_nها را تا جایی محاسبه می کنیم که

$$|f(x_N)| < \epsilon$$

به محض آنکه شرط فوق برقرار شد، x_N را به عنوان تقریب α می پذیریم.

ب- هر اختلاف دو x_n متوالی، مثلا x_{N-1} و x_N عدد کوچکی شود،

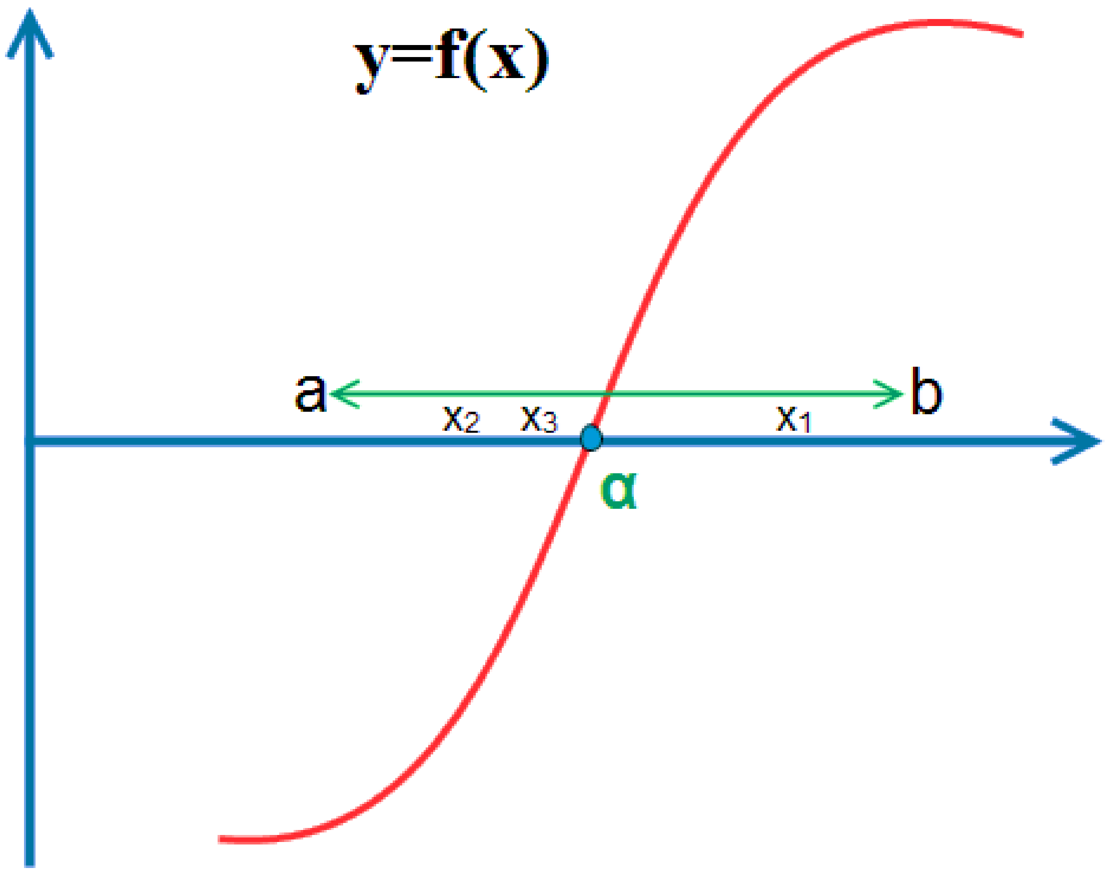
$$|x_N - x_{N-1}| < \epsilon$$

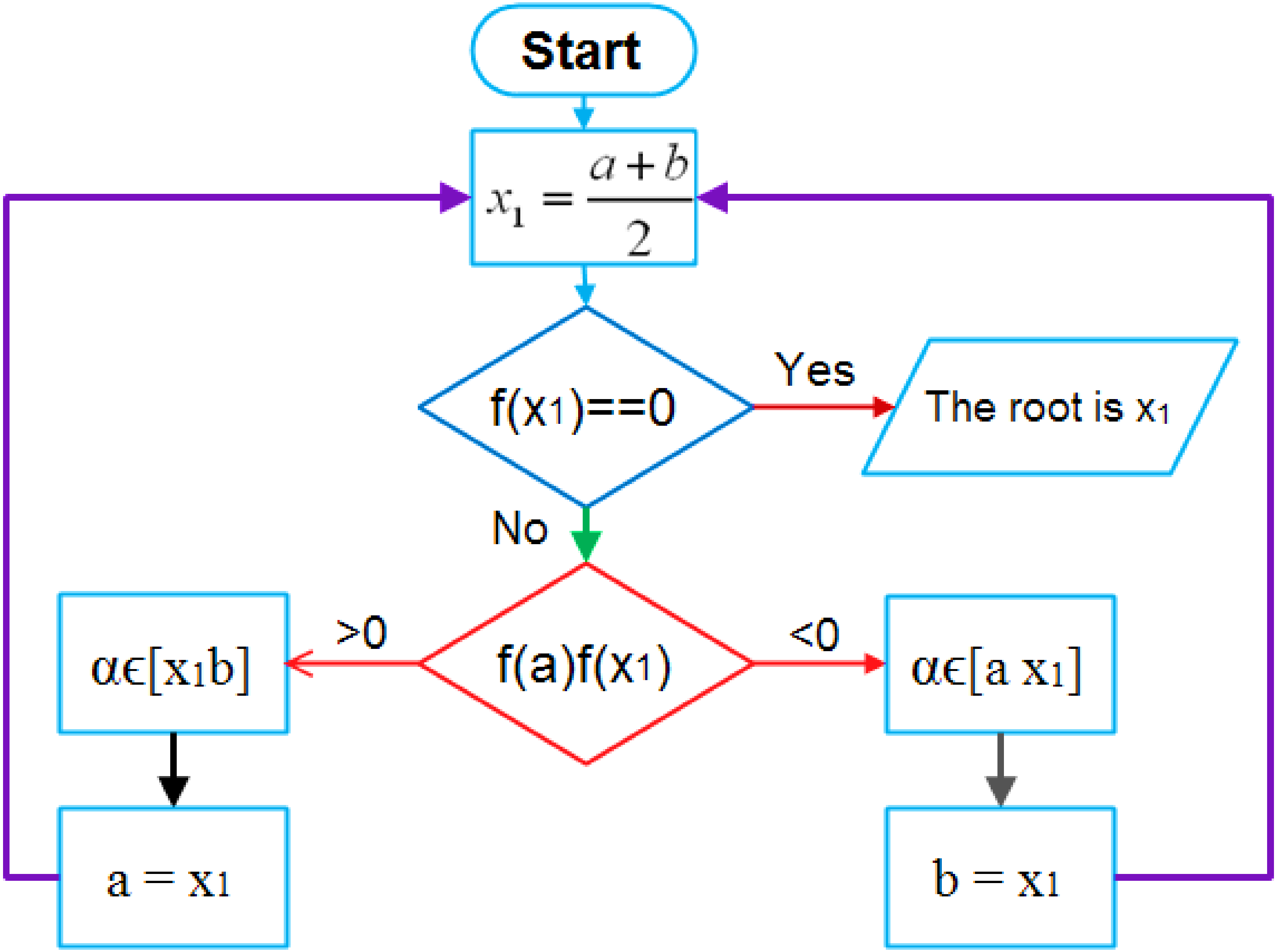
ج– هر گاه تعداد تکرار یا به عبارت دیگر n از یک عدد مشخص بیشتر شود، الگوریتم خاتمه یافته و x_n را به عنوان تقریب α می پذیریم.

روشهای حل عددی معادلات $f(x)=0$

روش دو بخشی یا تنصیف(bisection)

هرگاه شرایط ۱ تا ۳ برقرار باشد و α ریشه معادله $f(x)=0$ باشد، در این صورت نقطه $(\alpha\ 0)$ بر روی نمودار تابع $y=f(x)$ قرار دارد.




Start
$x_1 = \frac{a+b}{2}$
$f(x_1)==0$
Yes
The root is x_1
No
$f(a)f(x_1)$
>0
<0
$\alpha \epsilon [x_1 b]$
$\alpha \epsilon [a\ x_1]$
$a = x_1$
$b = x_1$

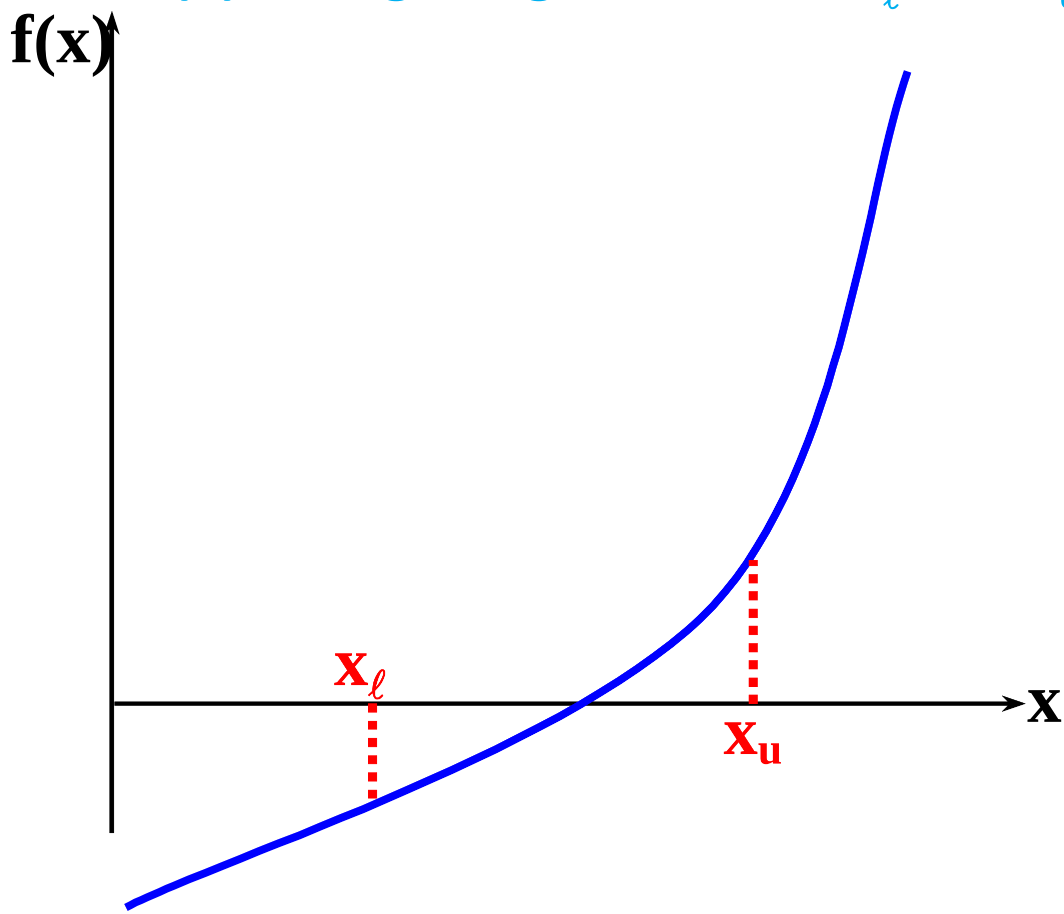
Step 1:

Choose x_ℓ and x_u as two guesses for the root such that

$f(x_\ell)\, f(x_u) < 0$,

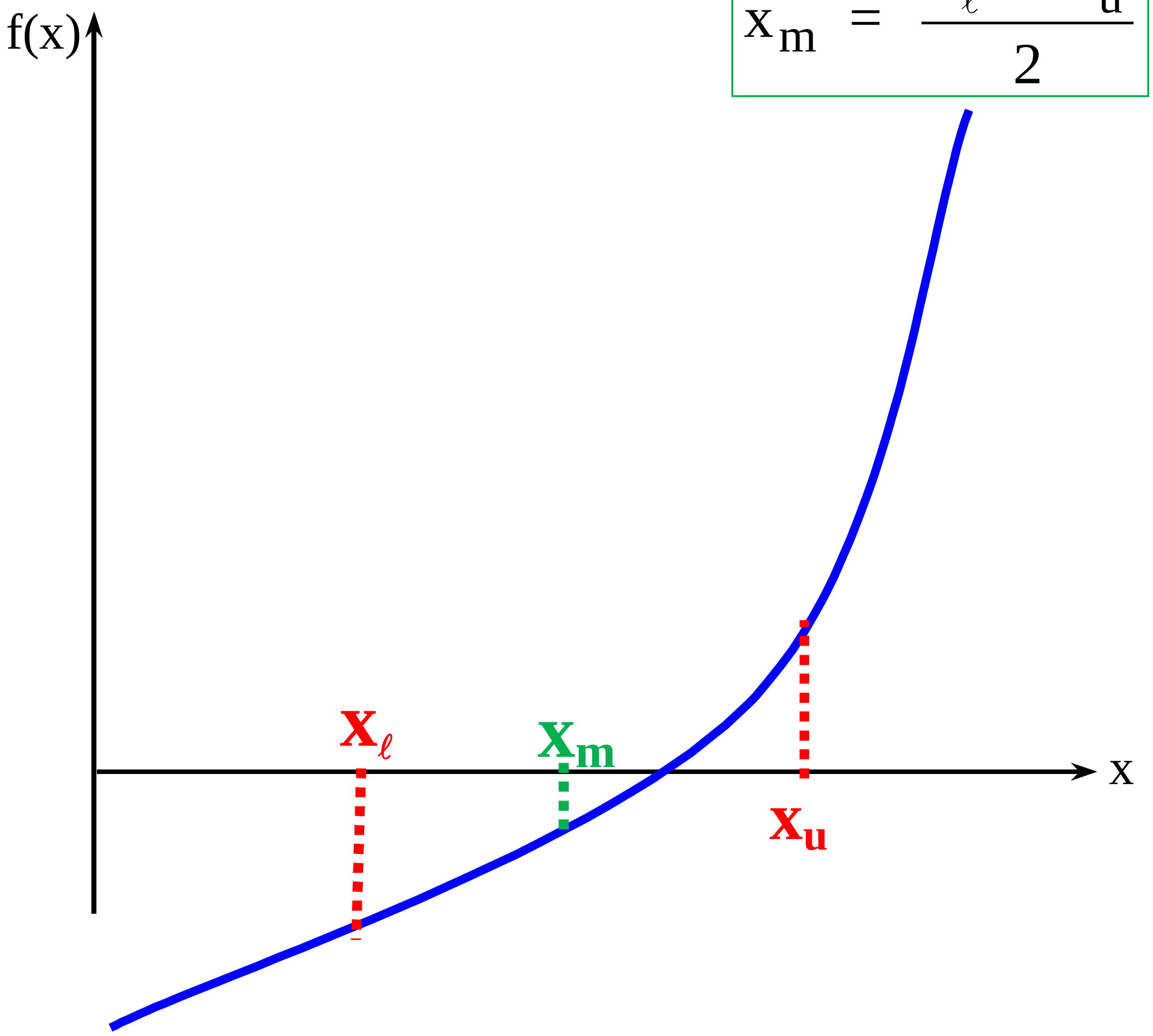
or in other words,

$f(x)$ changes sign between x_ℓ and x_u.



Step 2: Estimate the root, x_m of the equation f (x) = 0 as the mid point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$



Algorithm for Bisection Method

Step 3

Now check the following

a) **If** $f(x_l)f(x_m)<0$ **, then the root lies between x_ℓ and x_m; then $x_\ell = x_\ell$; $x_u = x_m$.**

b) **If** $f(x_l)f(x_m)>0$ **, then the root lies between x_m and x_u; then $x_\ell = x_m$; $x_u = x_u$.**

c) **If** $f(x_l)f(x_m)=0$ **; then the root is x_m. Stop the algorithm if this is true.**

Step 4

Find the new estimate of the root

$$x_m = \frac{x_\ell + x_u}{2}$$

Find the absolute relative approximate error

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

where

$$x_m^{old} = \text{previous estimate of root}$$

$$x_m^{new} = \text{current estimate of root}$$

Step 5

Compare the absolute relative approximate error $\left|\in_a\right|$ with the pre-specified error tolerance $\in_s$.

Is $\left|\in_a\right| > \in_s$?

Yes → **Go to Step 2 using new upper and lower guesses.**

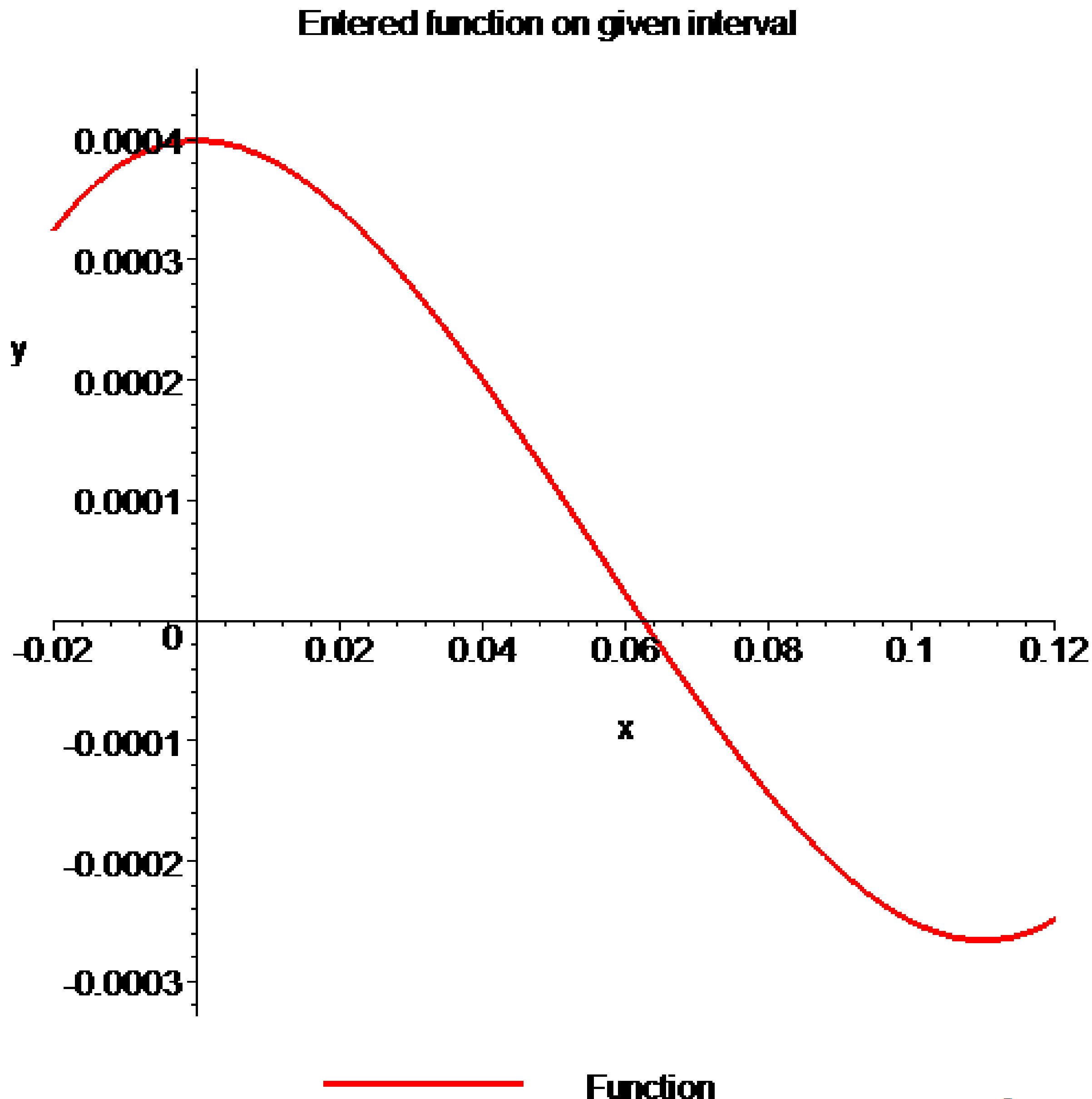
No → **Stop the algorithm**

Note one should also check whether the number of iterations is more than the maximum number of iterations allowed.

If so, one needs to terminate the algorithm and notify the user about it.

Example

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$



Let us assume

$$x_\ell = 0.00$$

$$x_u = 0.11$$

Check if the function changes sign between x_ℓ and x_u.

$$f(x_l) = f(0) = (0)^3 - 0.165(0)^2 + 3.993\times10^{-4} = 3.993\times10^{-4}$$

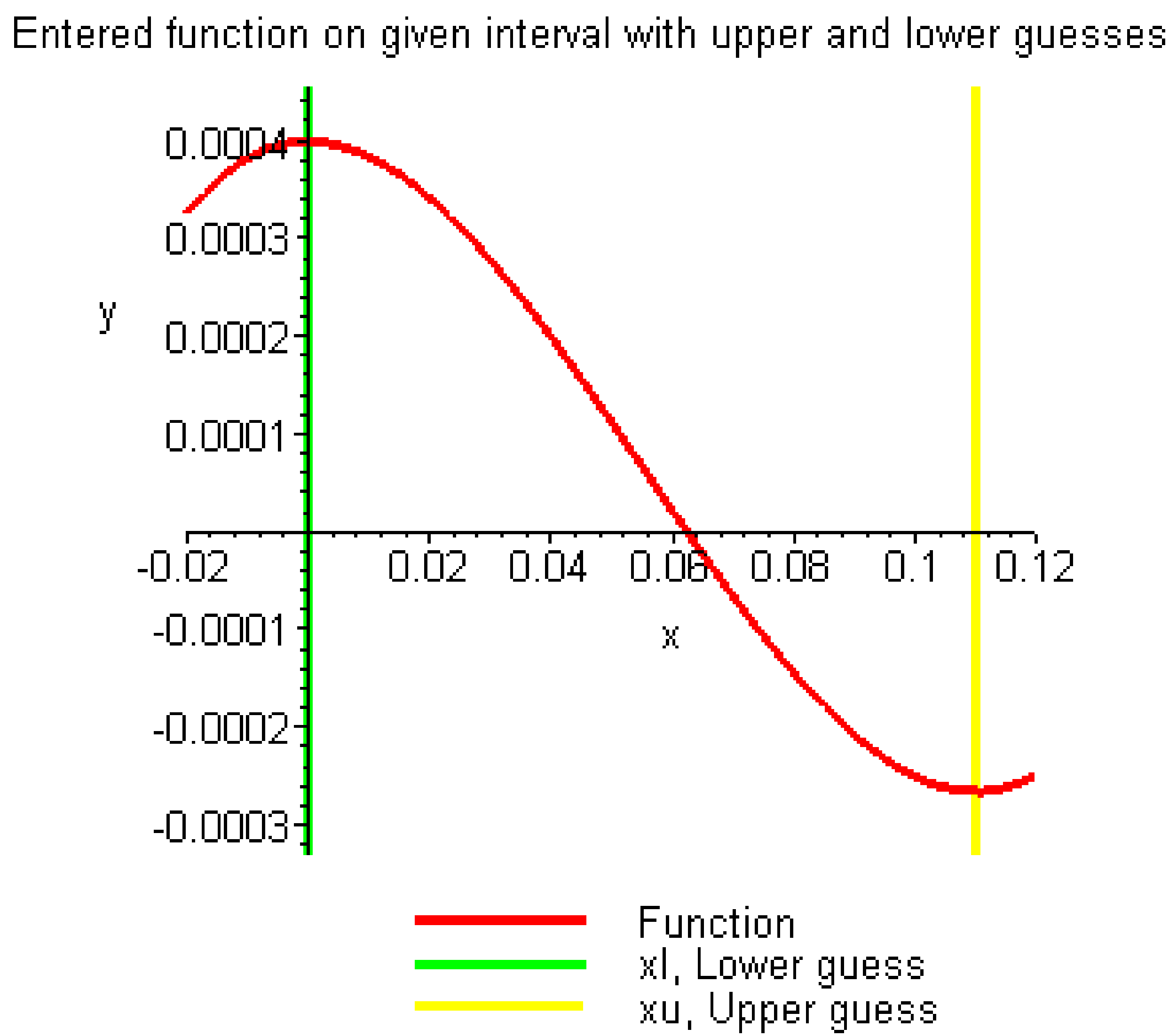
$$f(x_u) = f(0.11) = (0.11)^3 - 0.165(0.11)^2 + 3.993\times10^{-4} = -2.662\times10^{-4}$$

Hence

$$f(x_l)f(x_u) = f(0)f(0.11) = (3.993\times10^{-4})(-2.662\times10^{-4}) < 0$$

So there is at least on root between x_ℓ and x_u, that is between 0 and 0.11

Graph demonstrating sign change between initial limits



Iteration 1

The estimate of the root is $x_m = \frac{x_\ell + x_u}{2} = \frac{0+0.11}{2} = 0.055$

$$f(x_m) = f(0.055) = (0.055)^3 - 0.165(0.055)^2 + 3.993\times10^{-4} = 6.655\times10^{-5}$$

$$f(x_l)f(x_m) = f(0)f(0.055) = \left(3.993\times10^{-4}\right)\left(6.655\times10^{-5}\right) > 0$$

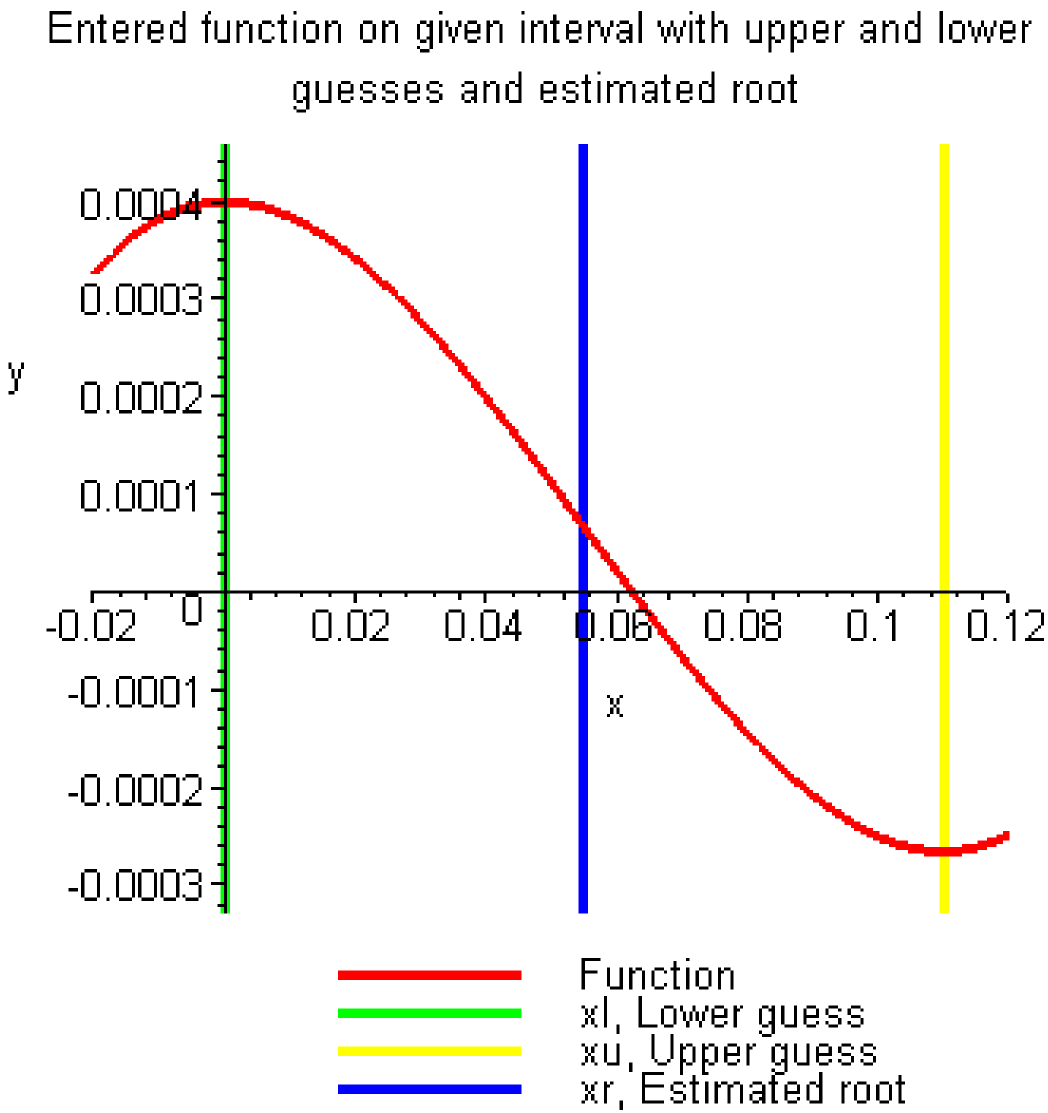
Hence the root is bracketed between x_m and x_u, that is, between 0.055 and 0.11.

So, the lower and upper limits of the new bracket are

$$x_l = 0.055,\ x_u = 0.11$$

At this point, the absolute relative approximate error $|\in_a|$ cannot be calculated as we do not have a previous approximation.

Estimate of the root for Iteration 1



Iteration 2

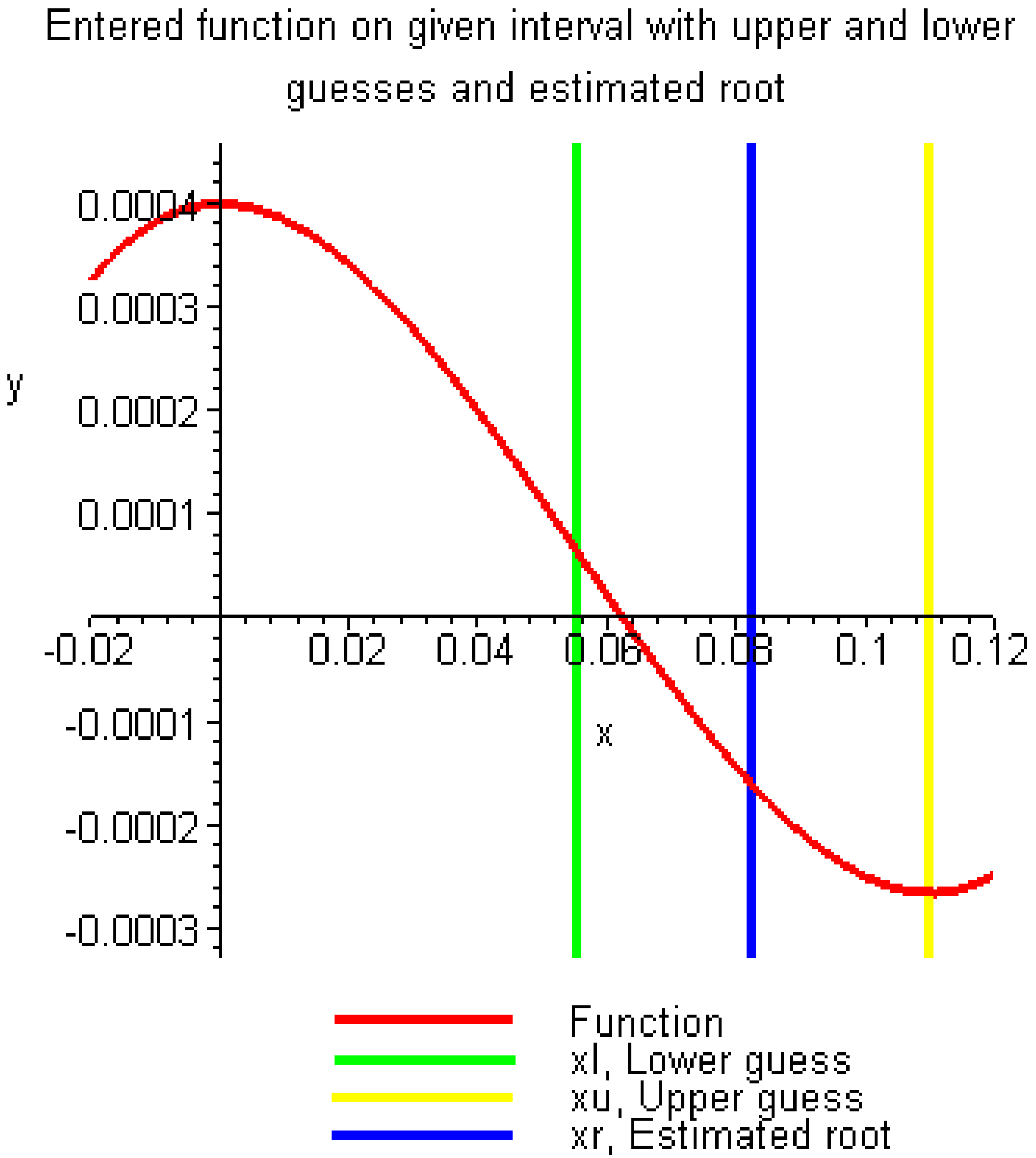
The estimate of the root is $x_m = \frac{x_\ell + x_u}{2} = \frac{0.055 + 0.11}{2} = 0.0825$

$$f(x_m) = f(0.0825) = (0.0825)^3 - 0.165(0.0825)^2 + 3.993\times10^{-4} = -1.622\times10^{-4}$$
$$f(x_l)f(x_m) = f(0)f(0.055) = (6.655\times10^{-5})(-1.622\times10^{-4}) < 0$$

Hence the root is bracketed between x_ℓ and x_m, that is, between 0.055 and 0.0825.
So, the lower and upper limits of the new bracket are

$$x_l = 0.055, \; x_u = 0.0825$$

Estimate of the root for Iteration 2



The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 2 is

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100 = \left|\frac{0.0825 - 0.055}{0.0825}\right| \times 100 = 33.333\%$$

None of the significant digits are at least correct in the estimate root of $x_m = 0.0825$

Because

the absolute relative approximate error is greater than 5%.

Iteration 3

The estimate of the root is $x_m = \frac{x_\ell + x_u}{2} = \frac{0.055 + 0.0825}{2} = 0.06875$

$$f(x_m) = f(0.06875) = (0.06875)^3 - 0.165(0.06875)^2 + 3.993\times10^{-4} = -5.563\times10^{-5}$$

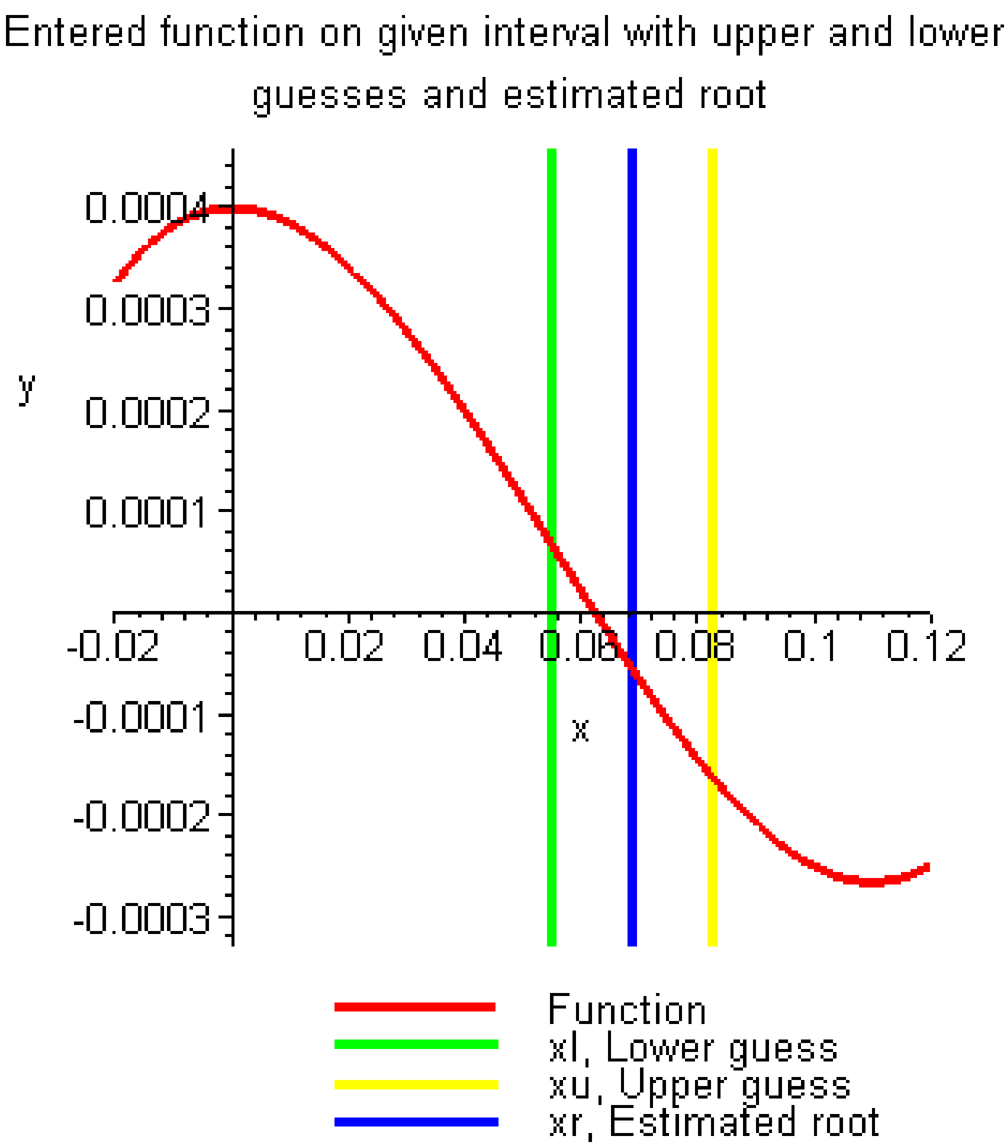
$$f(x_l)f(x_m) = f(0.055)f(0.06875) = (6.655\times10^{-5})(-5.563\times10^{-5}) < 0$$

Hence the root is bracketed between x_ℓ and x_m, **that is**, between 0.055 and 0.06875.

So, the lower and upper limits of the new bracket are

$$x_l = 0.055, \;\; x_u = 0.06875$$

Estimate of the root for Iteration 3

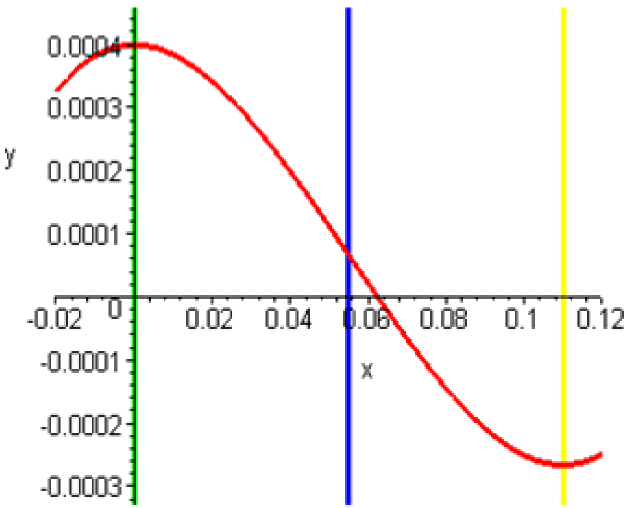


The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 3 is

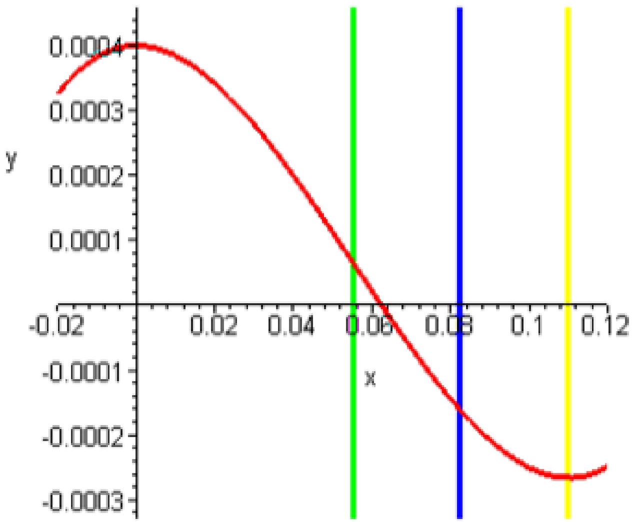
$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100 = \left|\frac{0.06875 - 0.0825}{0.06875}\right| \times 100 = 20\%$$

Still none of the significant digits are at least correct in the estimated root of the equation as the absolute relative approximate error is greater than 5%.

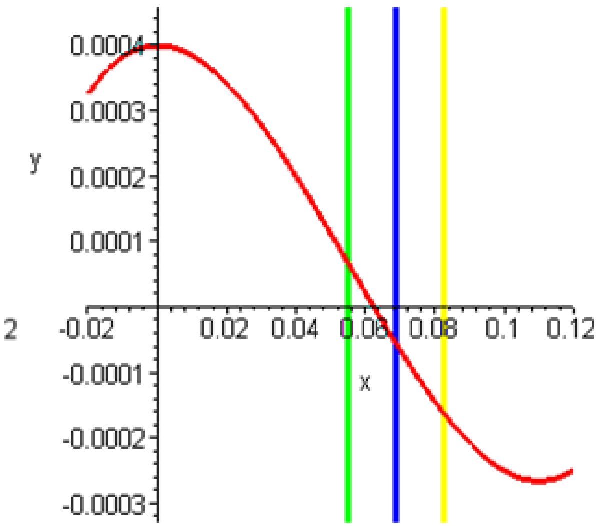
Iteration 1



Iteration 2



Iteration 3



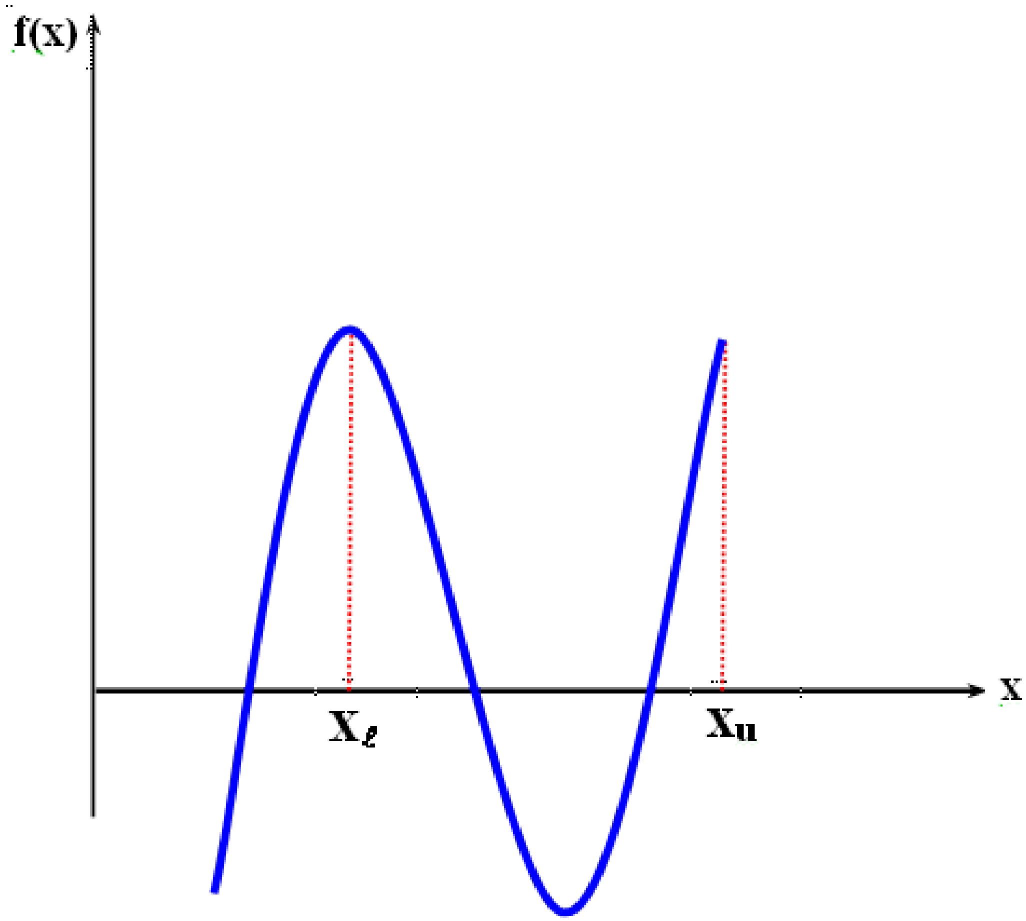
Root of f(x)=0 as function of number of iterations for bisection method.

Iteration	x_ℓ	x_u	x_m	$\lvert\epsilon_a\rvert$ %	$f(x_m)$
1	0.00000	0.11	0.055	----------	6.655×10^{-5}
2	0.055	0.11	0.0825	33.33	-1.622×10^{-4}
3	0.055	0.0825	0.06875	20.00	-5.563×10^{-5}
4	0.055	0.06875	0.06188	11.11	4.484×10^{-6}
5	0.06188	0.06875	0.06531	5.263	-2.593×10^{-5}
6	0.06188	0.06531	0.06359	2.702	-1.0804×10^{-5}
7	0.06188	0.06359	0.06273	1.370	-3.176×10^{-6}
8	0.06188	0.06273	0.0623	0.6897	6.497×10^{-7}
9	0.0623	0.06273	0.06252	0.3436	-1.265×10^{-6}
10	0.0623	0.06252	0.06241	0.1721	-3.0768×10^{-7}

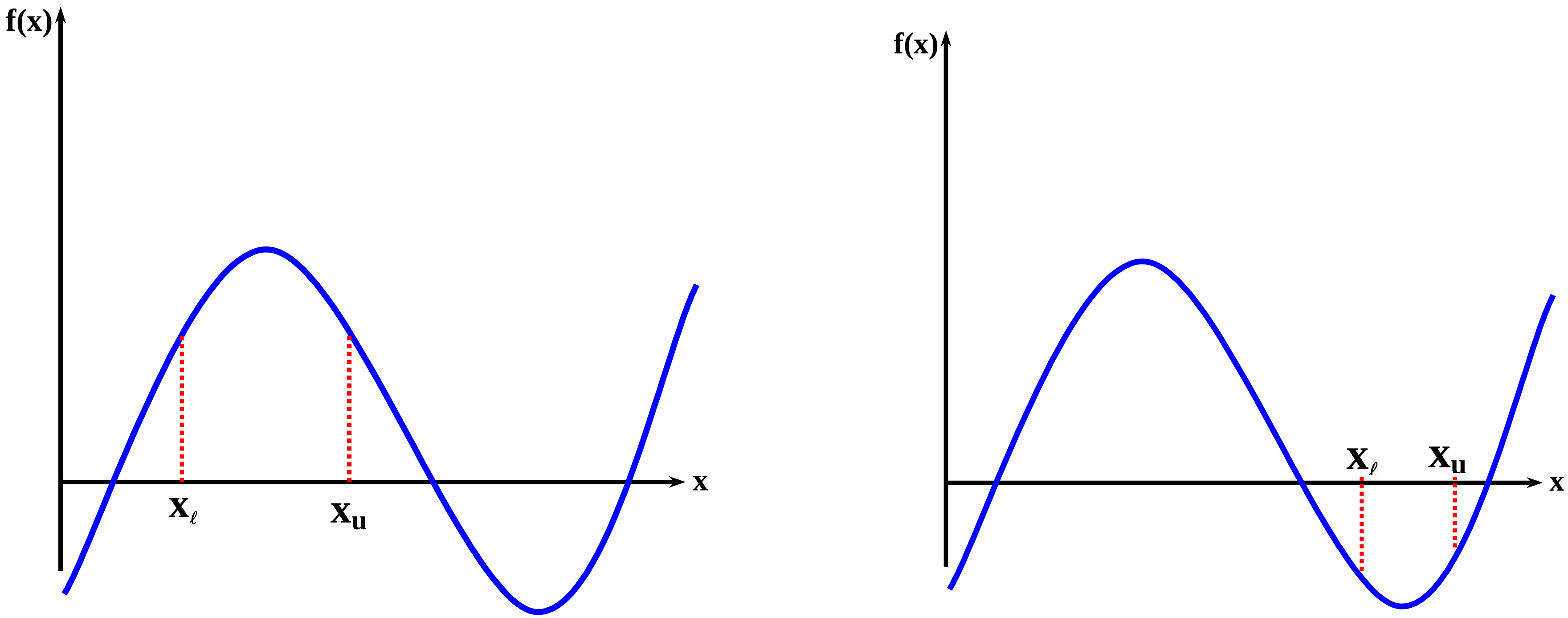
تقریبی از ریشه معادله $f(x) = 3x - e^{-x} = 0$ را که در فاصله (۰/۲۷ ۰/۲۵) قرار دارد، با سه رقم اعشار به دست آورید،

به طوریکه داشته باشیم $|f(x_n)| < 0.001$ که x_n تقریب ریشه در تکرار nام است.

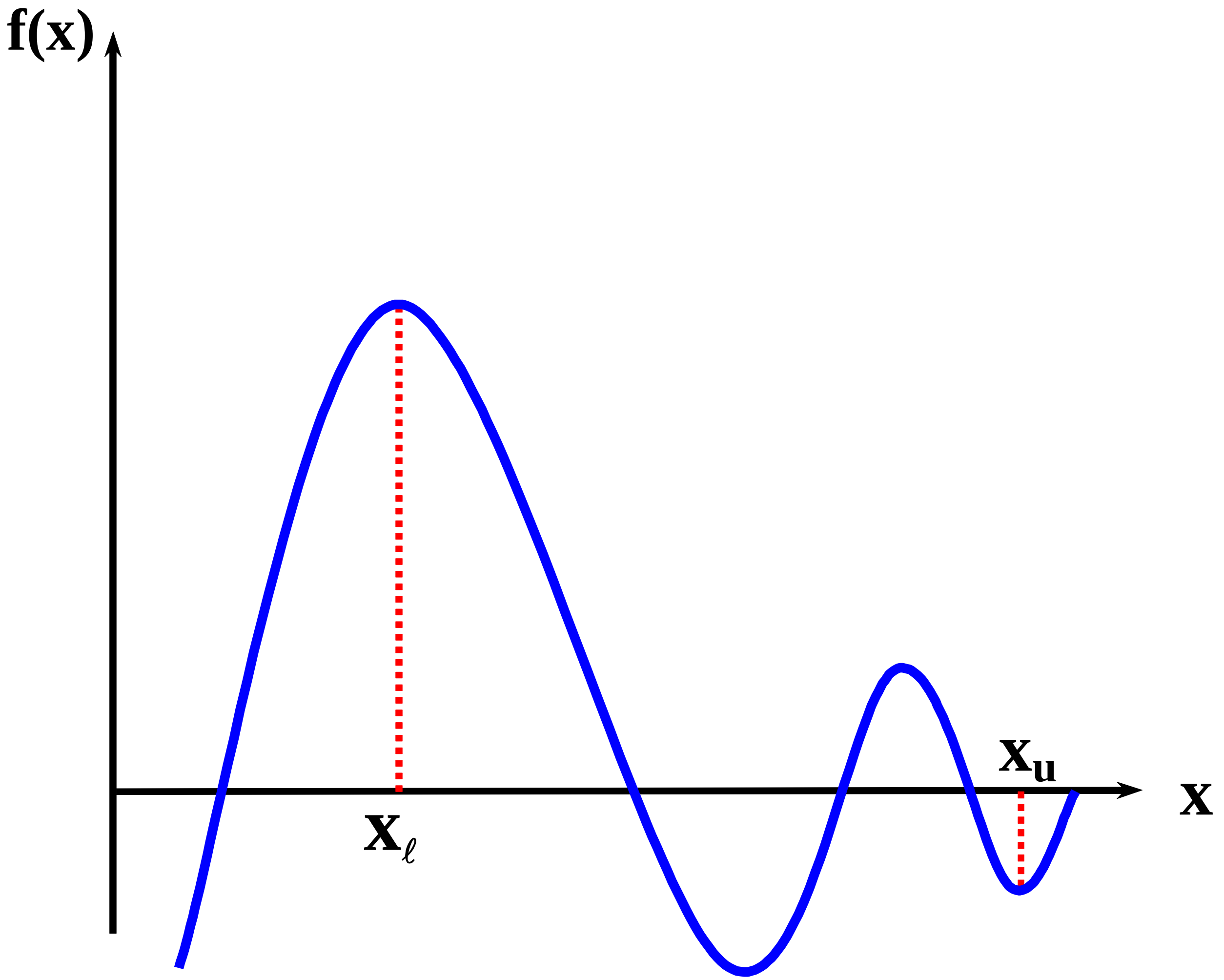
n	a	b	x_n	$f(a)\,f(x_n)$	$f(x_n)$
1	0.25	0.27	0.26	-	0.0089
2	0.25	0.26	0.255	+	-0.0099
3	0.255	0.26	0.2575	+	-0.0005



If function $f(x)$ does not change sign between two points, roots of the equation $f(x)=0$ may still exist between the two points.



If the function $f(x)$ does not change sign between two points, there may not be any roots for the equation $f(x)=0$ between the two points.



If the function $f(x)$ changes sign between two points, more than one root for the equation $f(x)=0$ may exist between the two points.

Advantages

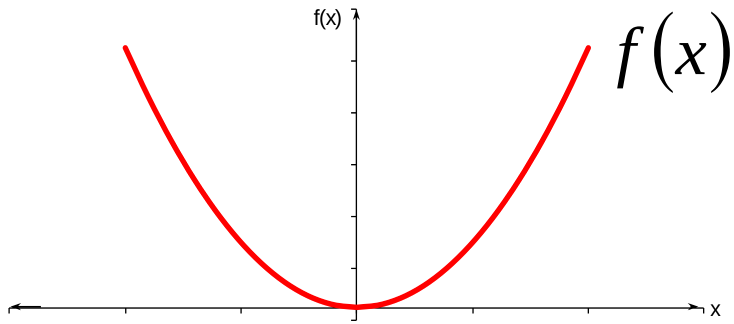
- Always convergent
- The root bracket gets halved with each iteration - guaranteed.

Drawbacks

- Slow convergence
- If one of the initial guesses is close to the root, the convergence is slower

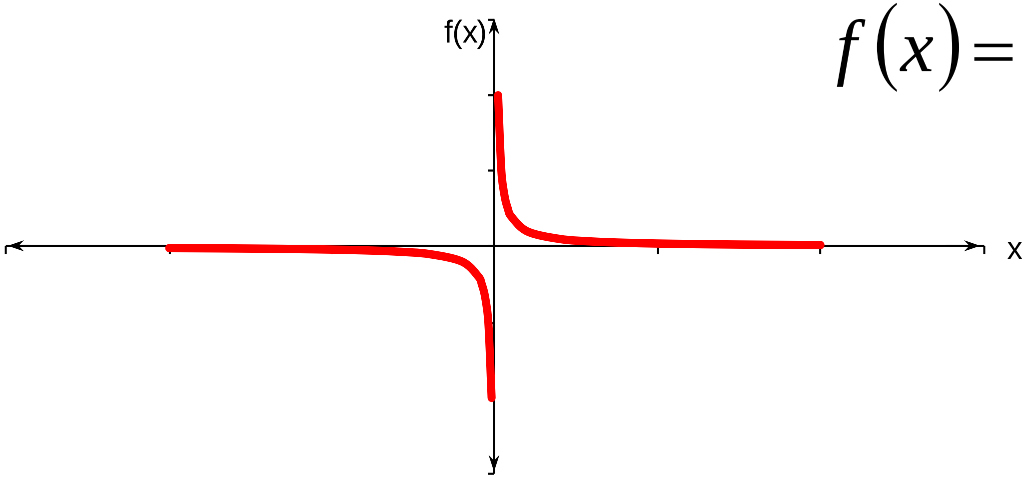
Drawbacks

- **If a function f(x) is such that it just touches the x-axis it will be unable to find the lower and upper guesses.**



$$f(x) = x^2$$

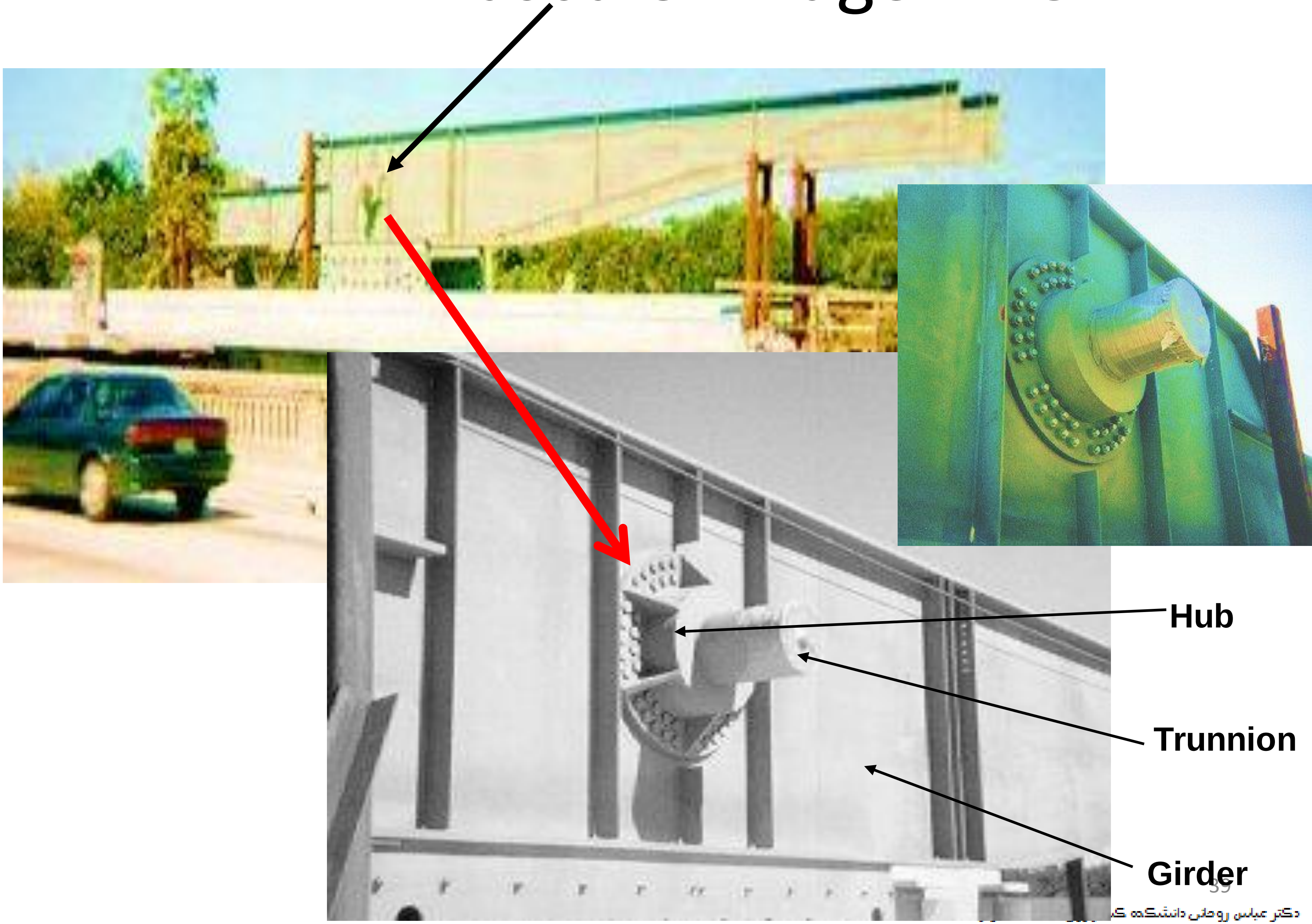
- **Function changes sign but root does not exist**



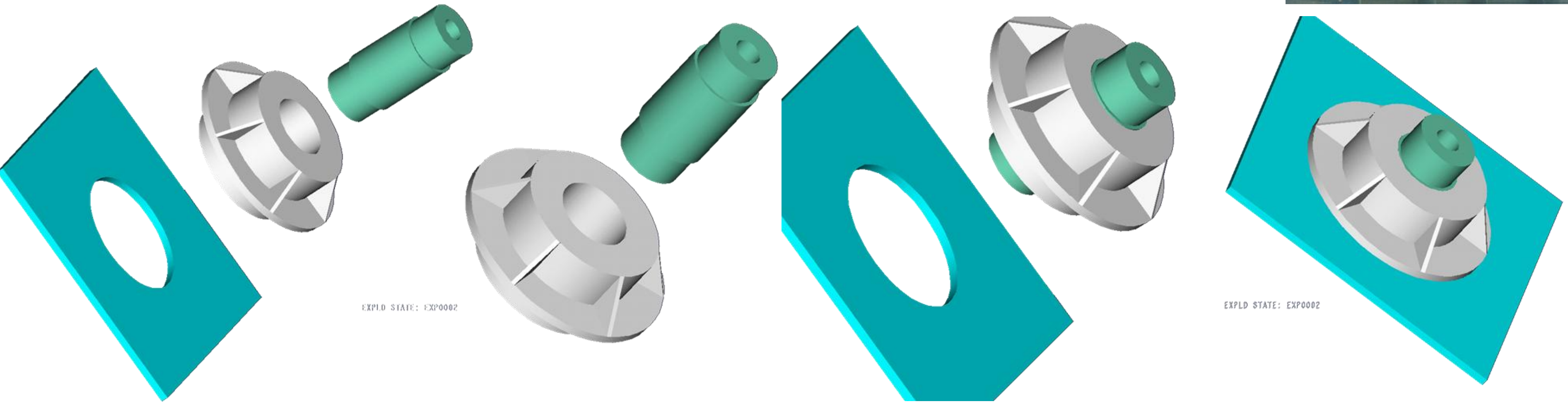
$$f(x) = \frac{1}{x}$$

مثال کاربردی مکانیکی

Bascule Bridge THG



Trunnion-Hub-Girder Assembly Procedure



Step1. Trunnion immersed in dry-ice/alcohol

Step2. Trunnion warm-up in hub

Step3. Trunnion-Hub immersed in dry-ice/alcohol

Step4. Trunnion-Hub warm-up into girder

A trunnion has to be cooled before it is shrink fitted into a steel hub.

The equation that gives the temperature T_f to which the trunnion has to be cooled to obtain the desired contraction is given by

$$f(T_f) = -0.50598\times10^{-10}T_f^3 + 0.38292\times10^{-7}T_f^2 + 0.74363\times10^{-4}T_f + 0.88318\times10^{-2} = 0$$

Use the bisection method of finding roots of equations to find the temperature T_f to which the trunnion has to be cooled.

Conduct three iterations to estimate the root of the above. Find the absolute relative approximate error at the end of each iteration and the number of significant digits at least correct at the end of each iteration.

Solution

From the designer's records for the previous bridge, the temperature to which the trunnion was cooled was $-108°F$.

Hence assuming the temperature to be between $-150°F$ and $-100°F$

we have

$$T_{f,\ell} = -150\,°F \qquad T_{f,u} = -100\,°F$$

Check if the function changes sign between $T_{f,\ell}$ and $T_{f,u}$

$$f\left(T_{f,\ell}\right) = f\left(-150\right) = -1.2903 \times 10^{-3}$$

$$f\left(T_{f,u}\right) = f\left(-100\right) = 1.8290 \times 10^{-3}$$

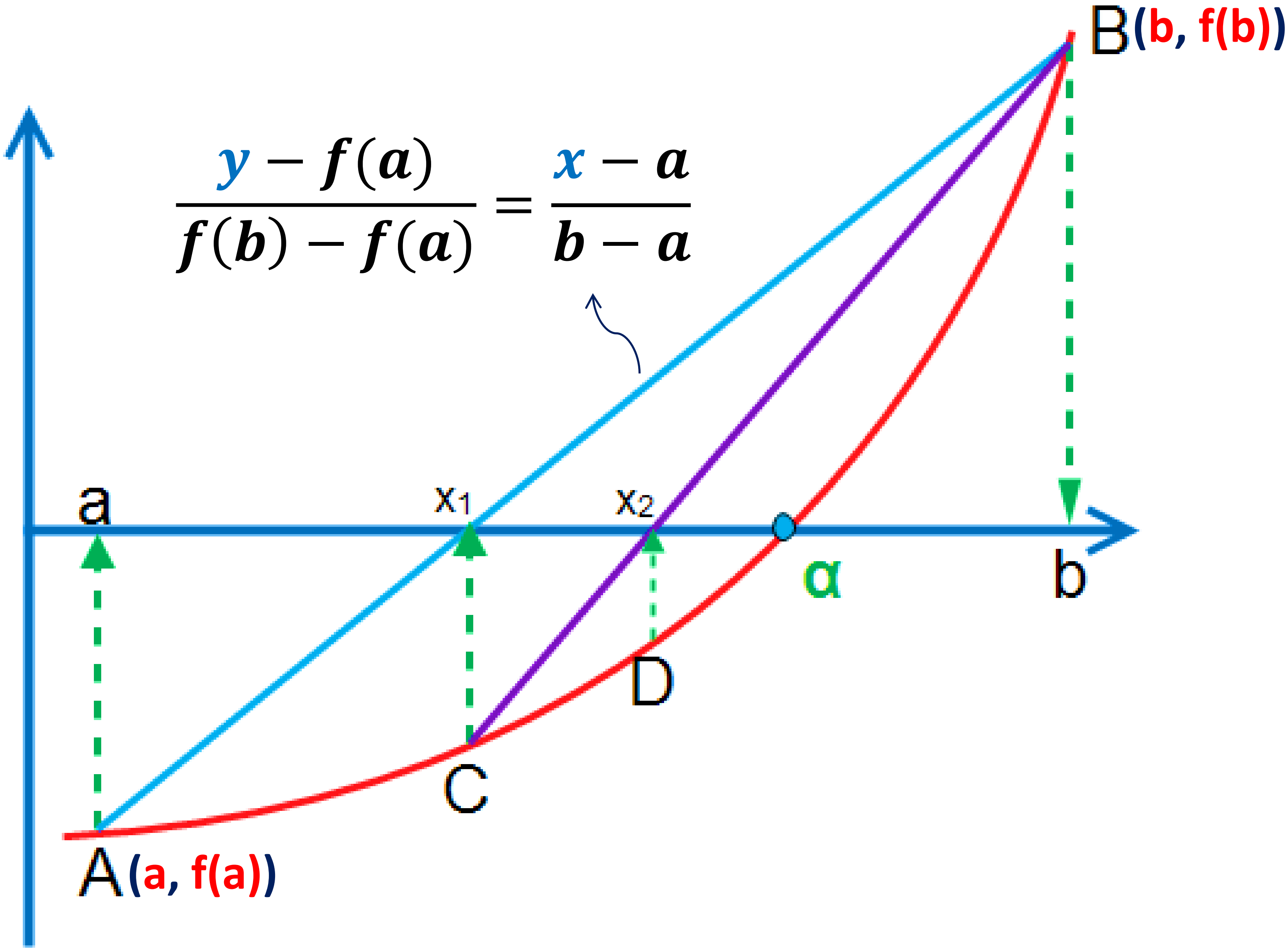
Hence $$f\left(T_{f,\ell}\right) f\left(T_{f,u}\right) = f\left(-150\right) f\left(-100\right) < 0$$

So there is at least one root between -150 and -100.

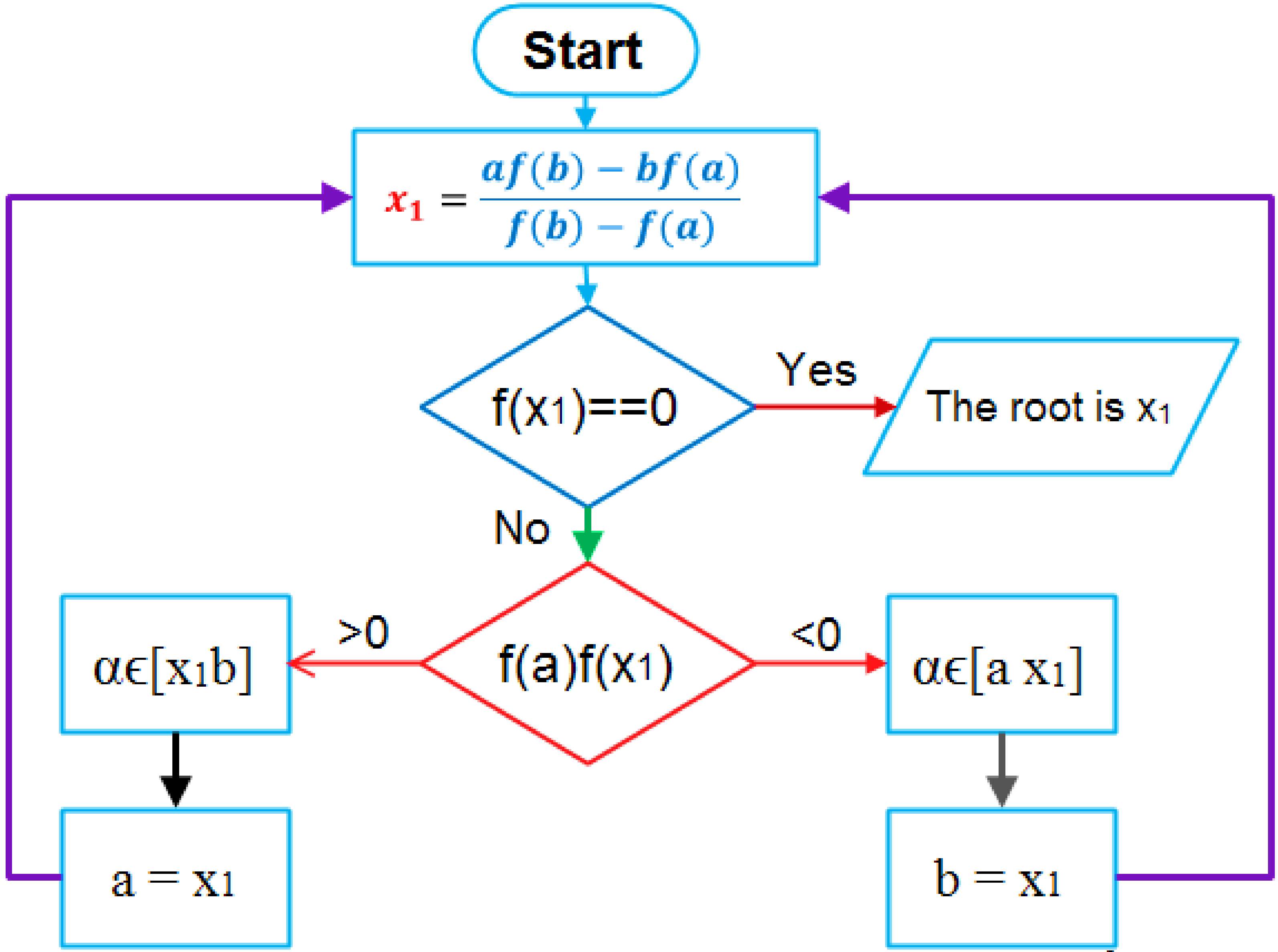
Iteration	$T_{f,\ell}$	$T_{f,u}$	$T_{f,m}$	$\lvert \epsilon_a \rvert \%$	$f(T_{f,m})$
1	−150	−100	−125	---------	2.3356×10^{-4}
2	−150	−125	−137.5	9.0909	-5.3762×10^{-4}
3	−137.5	−125	−131.25	4.7619	-1.5430×10^{-4}
4	−131.25	−125	−128.13	2.4390	3.9065×10^{-5}
5	−131.25	−128.13	−129.69	1.2048	-5.7760×10^{-5}
6	−129.69	−123.13	−128.91	0.60606	-9.3826×10^{-6}
7	−128.91	−123.13	−128.52	0.30395	1.4838×10^{-5}
8	−128.91	−128.52	−128.71	0.15175	2.7228×10^{-6}
9	−128.91	−128.71	−128.81	0.075815	-3.3305×10^{-6}
10	−128.81	−128.71	−128.76	0.037922	-3.0396×10^{-7}

روش نابجایی

$$\frac{y-f(a)}{f(b)-f(a)}=\frac{x-a}{b-a}$$



$$\frac{0 - f(a)}{f(b) - f(a)} = \frac{x_1 - a}{b - a} \Rightarrow x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$



تمرین: مسئله زیر را به کمک روش نابجایی حل کنید و با روش دو بخشی مقایسه کنید؟.

You are making a bookshelf to carry books that range from 8½" to 11" in height and would take up 29" of space along the length. The material is wood having a Young's Modulus of 3.667 Msi, thickness of 3/8" and width of 12".

You want to find the maximum vertical deflection of he bookshelf. The vertical deflection of the shelf is given by

$$v(x) = 0.42493 \times 10^{-4} x^3 - 0.13533 \times 10^{-8} x^5 - 0.66722 \times 10^{-6} x^4 - 0.018507x$$

where x is the position along the length of the beam. Hence to find the maximum deflection we need to find where

$$f(x) = \frac{dv}{dx} = 0$$

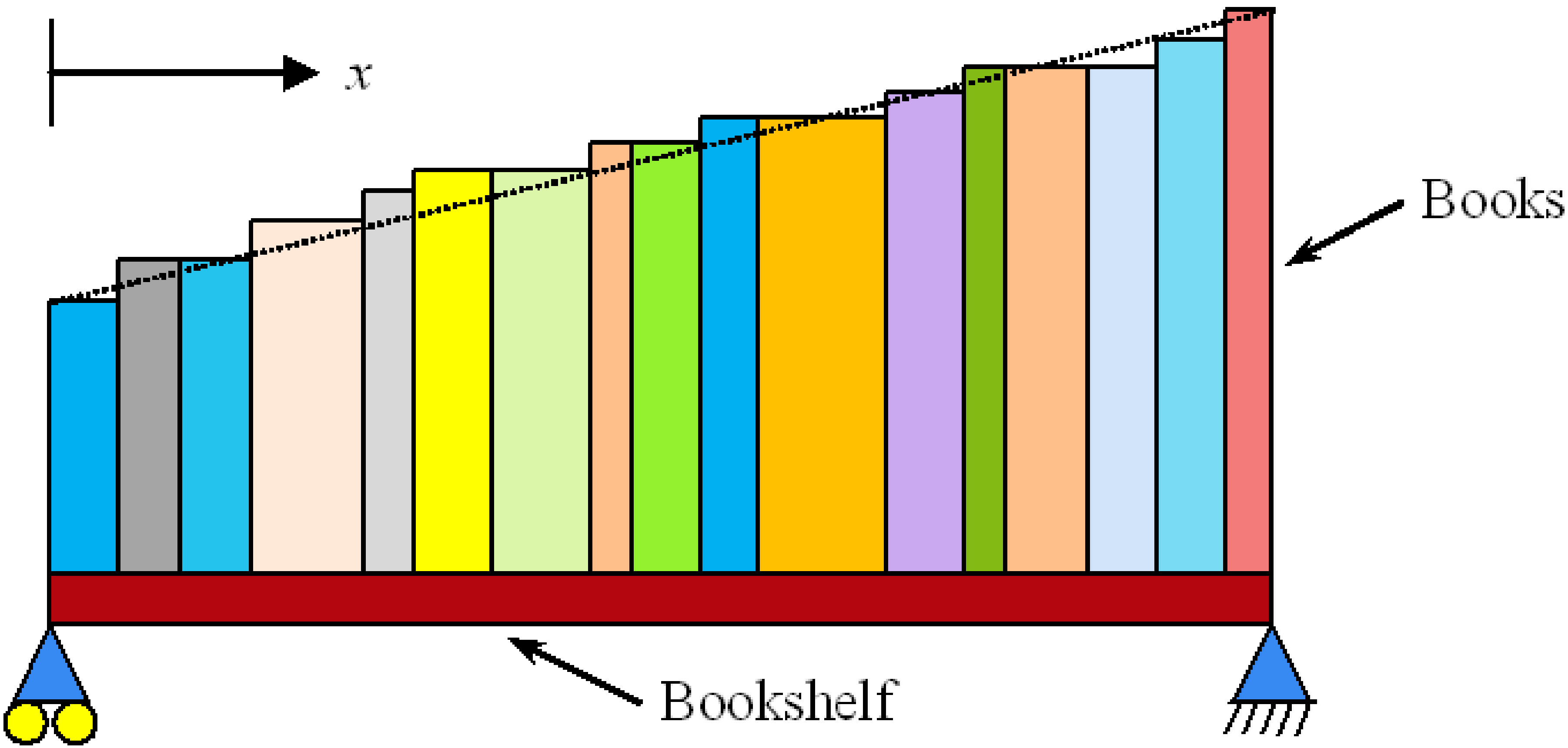
The equation that gives the position x where the deflection is maximum is given by

$$-0.67665 \times 10^{-8} x^4 - 0.26689 \times 10^{-5} x^3 + 0.12748 \times 10^{-3} x^2 - 0.018507 = 0$$

Use the **bisection** method of finding roots of equations to find the position *x* where the deflection is maximum.

Conduct three iterations to estimate the root of the above equation.

Find the absolute relative approximate error at the end of each iteration at the end of each iteration.



Solution

From the physics of the problem, the maximum deflection would be between $x = 0$ and $x = L$, where

$$L = \text{Length of the bookshelf},$$

That is

$$0 \le x \le L$$

$$0 \le x \le 29$$

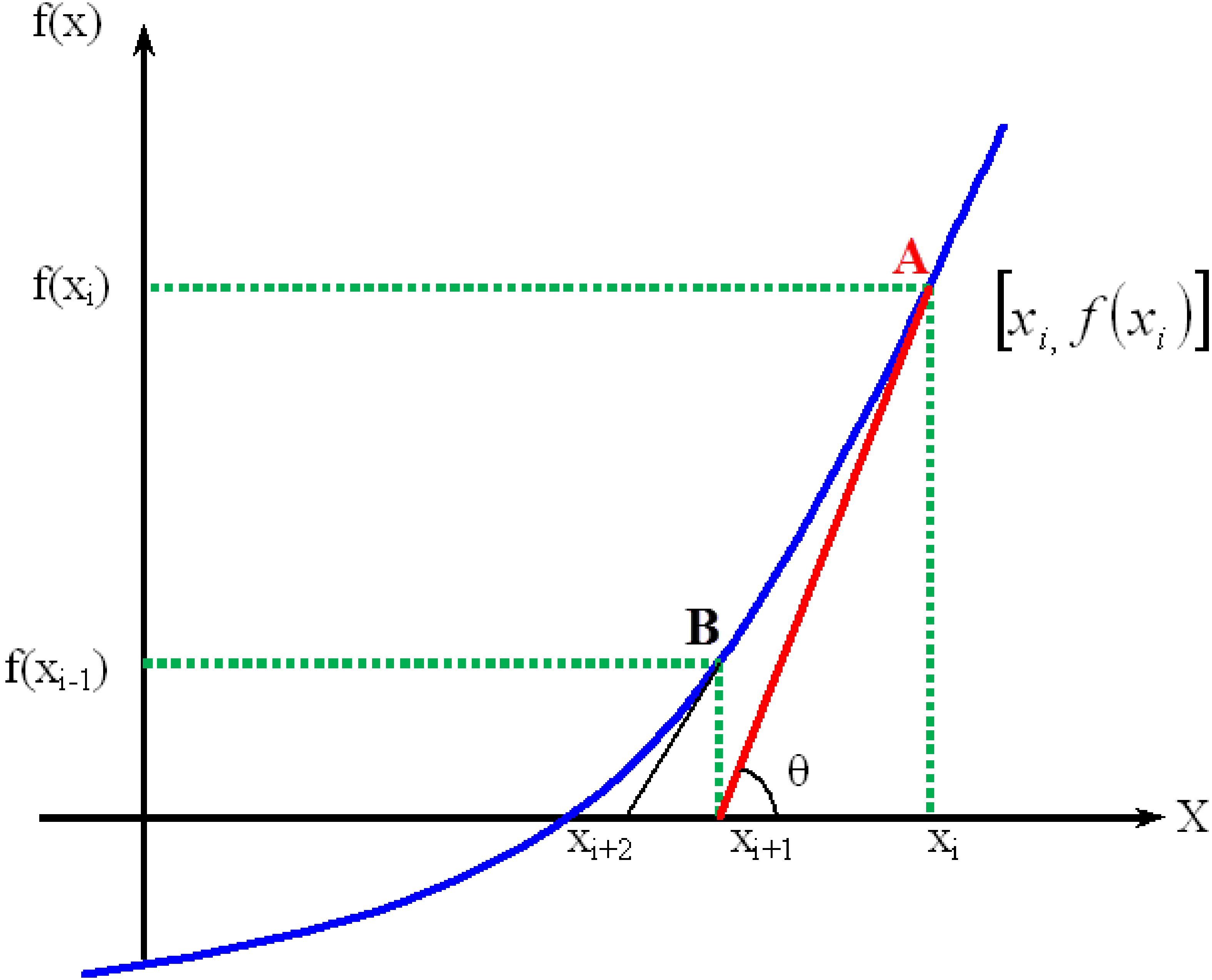
Let us assume

$$x_\ell = 0,\ x_u = 29$$

Root of *f(x)=0* as a function of the number of iterations for bisection method.

Iteration	x_ℓ	x_u	x_m	$\lvert\in_a\rvert\%$	$f(x_m)$
1	0	29	14.5	----------	-1.3992×10^{-4}
2	14.5	29	21.75	33.333	0.012824
3	14.5	21.75	18.125	20	6.7502×10^{-3}
4	14.5	18.125	16.313	11.111	3.3509×10^{-3}
5	14.5	16.313	15.406	5.8824	1.6099×10^{-3}
6	14.5	15.406	14.953	3.0303	7.3521×10^{-4}
7	14.5	14.953	14.727	1.5385	2.9753×10^{-4}
8	14.5	14.727	14.613	0.77519	7.8708×10^{-5}
9	14.5	14.613	14.557	0.38911	-3.0688×10^{-5}
10	14.557	14.613	14.585	0.19417	2.4009×10^{-5}

Geometrical illustration of the Newton-Raphson method.



از نقطه $A(x_i,f(x_i))$ واقع بر منحنی $y=f(x)$ مماسی بر منحنی رسم می کنیم.

با داشتن x_i برای تعیین x_{i+1}، بایستی معادله خط مماس بر منحنی $y=f(x)$ را در نقطه $A(x_i,f(x_i))$ بنویسیم و محل تلاقی آن را با محور xها تعیین کنیم.

ضریب زاویه این خط مماس $m = f'(x_i)$ است. بنابراین داریم:

$$y - f(x_i) = f'(x_i)(x - x_i)$$

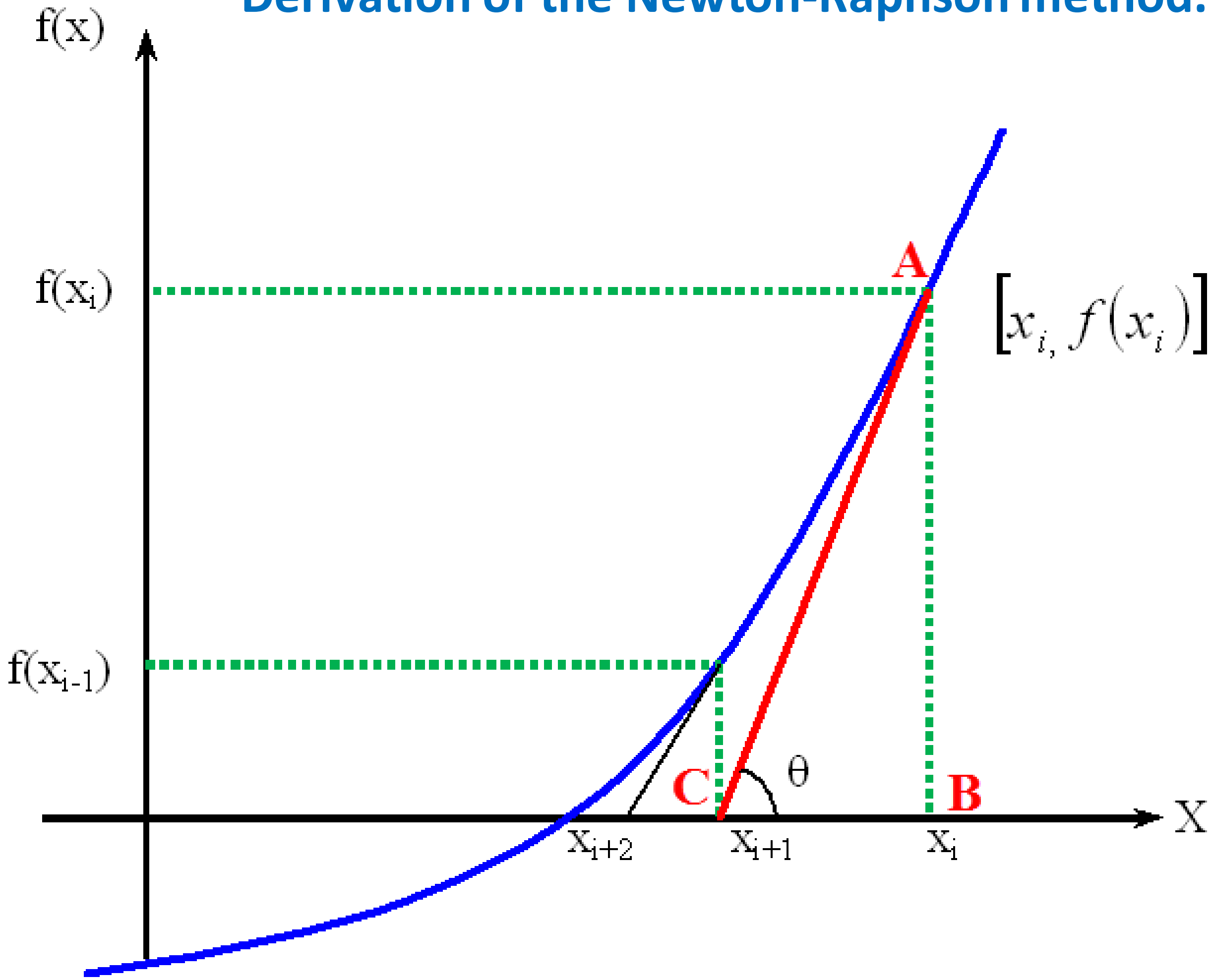
محل تلاقی این خط با محور طولها را $(x_{i+1},0)$ می گیریم:

$$0 - f(x_i) = f'(x_i)(x_{i+1} - x_i)$$

و اگر $f'(x_i) \neq 0$ باشد آنگاه داریم:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Derivation of the Newton-Raphson method.



$$\tan(\alpha) = \frac{AB}{BC} \Rightarrow f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}} \Rightarrow x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Algorithm for Newton-Raphson Method

Step 1: Evaluate $f'(x)$ symbolically.

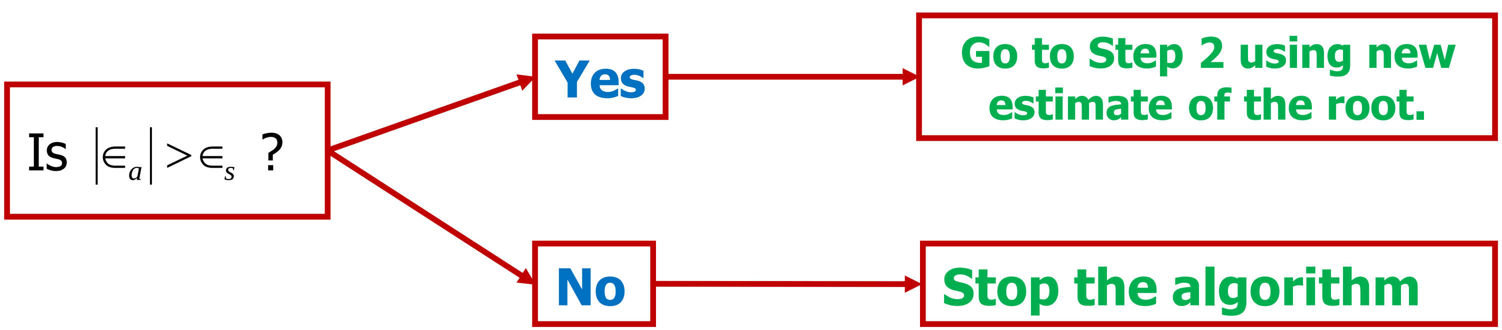
Step 2: Use an initial guess of the root, x_i , to estimate the new value of the root, x_{i+1}, as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

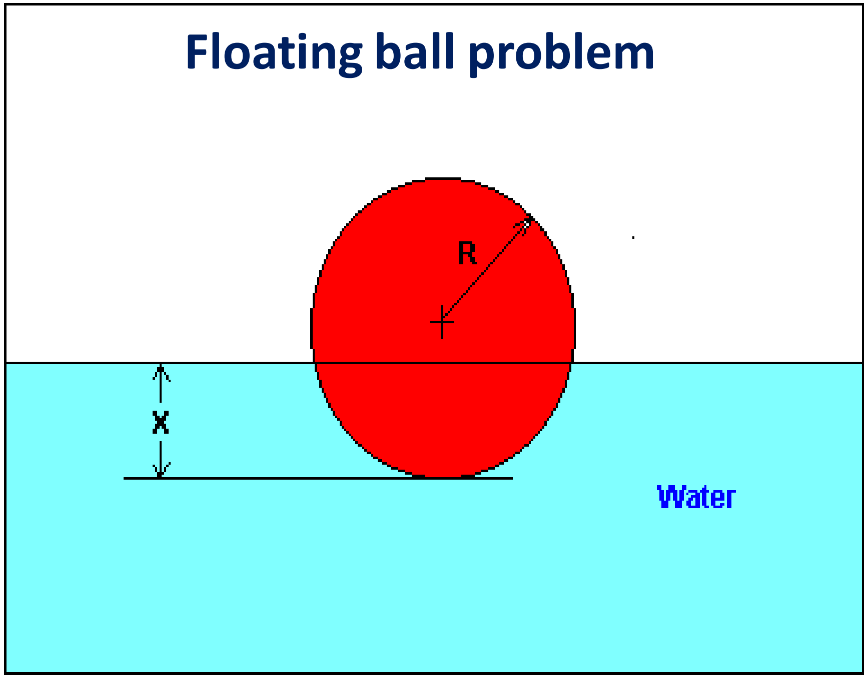
Step 3 : Find the absolute relative approximate error $|\in_a|$ as

$$|\in_a| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100$$

Step 4: Compare the absolute relative approximate error with the pre-specified relative error tolerance $\in_s$



The floating ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the depth to which the ball is submerged when floating in water.



The equation that gives the depth *x* in meters to which the ball is submerged under water is given by

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$

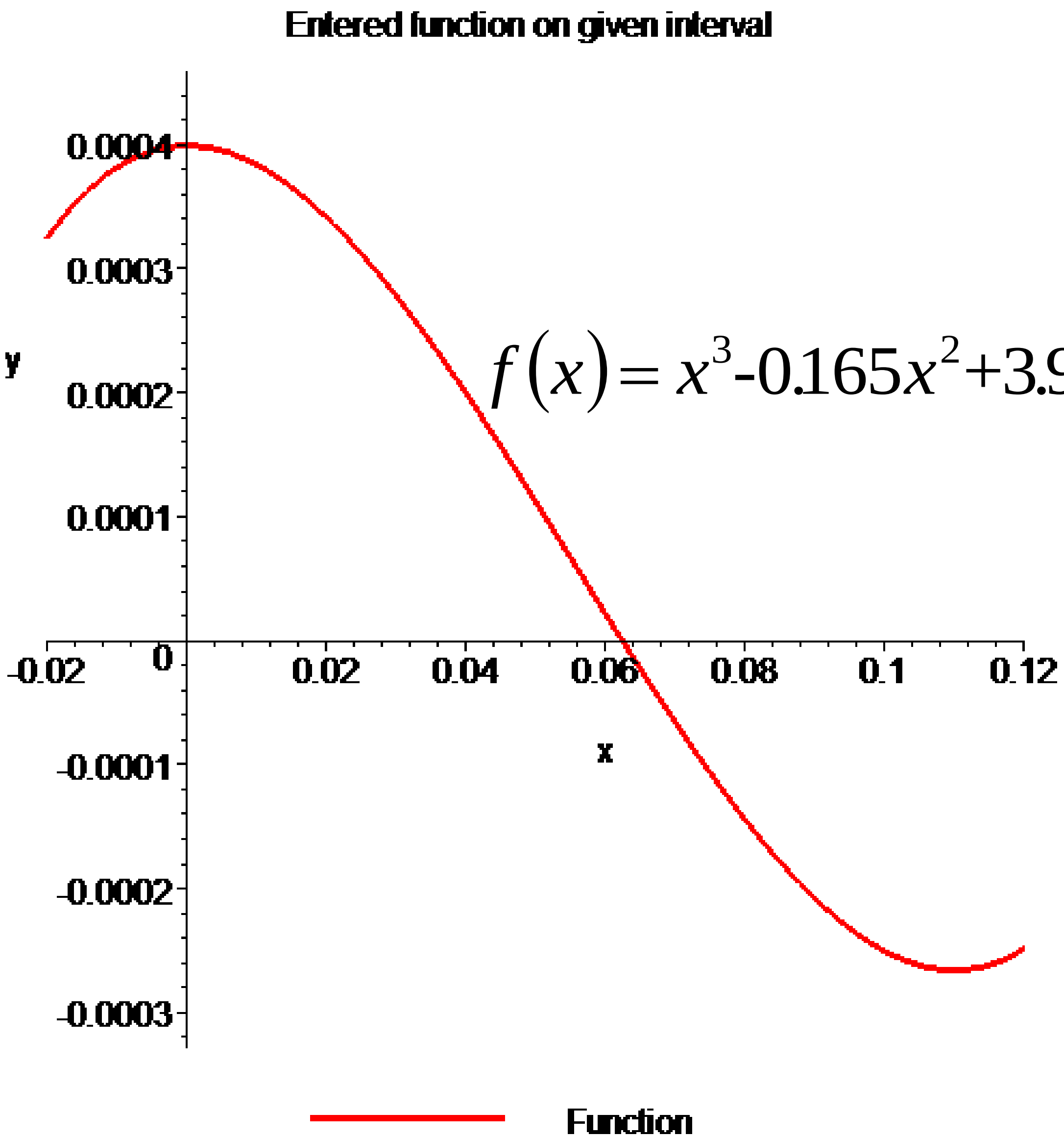
Use the Newton's method of finding roots of equations to find

a) the depth 'x' to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation.

b) The absolute relative approximate error at the end of each iteration

To aid in the understanding of how this method works to find the root of an equation, the graph of f(x) is shown to the right, where

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$



Solve for $f'(x)$

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4} \quad \Rightarrow \quad f'(x) = 3x^2 - 0.33x$$

Let us assume the initial guess of the root of *f(x)=0* is $x_0 = 0.05\text{m}$. This is a reasonable guess (discuss why $x = 0$ and $x = 0.11\text{m}$ are not good choices) as the extreme values of the depth *x* would be 0 and the diameter (0.11 m) of the ball.

Iteration 1

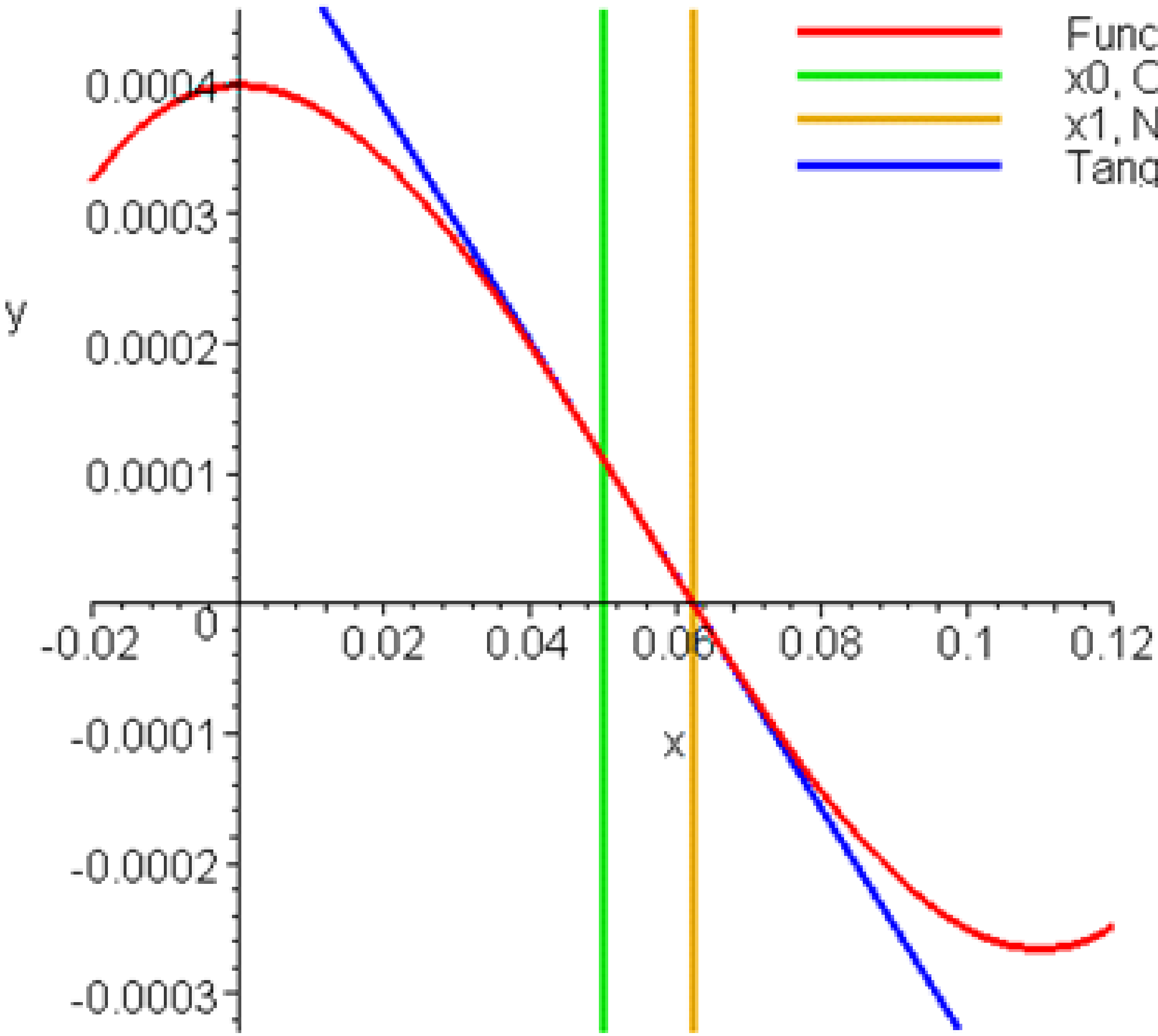
The estimate of the root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.05 - \frac{(0.05)^3 - 0.165(0.05)^2 + 3.993\times10^{-4}}{3(0.05)^2 - 0.33(0.05)}$$

$$= 0.05 - \frac{1.118\times10^{-4}}{-9\times10^{-3}} = 0.05 - (-0.01242) = 0.06242$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 1 is

$$|\in_a| = \left|\frac{x_1 - x_0}{x_1}\right| \times 100 = \left|\frac{0.06242 - 0.05}{0.06242}\right| \times 100 = 19.90\%$$



Iteration 2

The estimate of the root is

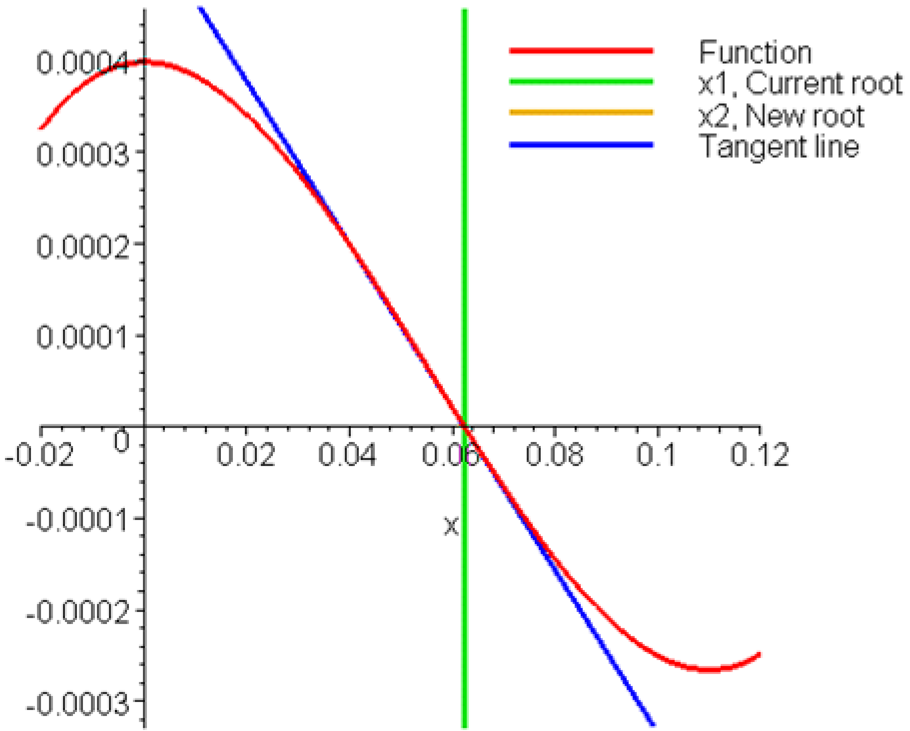
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.06242 - \frac{(0.06242)^3 - 0.165(0.06242)^2 + 3.993\times10^{-4}}{3(0.06242)^2 - 0.33(0.06242)}$$

$$= 0.06242 - \frac{-3.97781\times10^{-7}}{-8.90973\times10^{-3}} = 0.06242 - \left(4.4646\times10^{-5}\right) = 0.06238$$

$$|\in_a| = \left|\frac{x_2 - x_1}{x_2}\right| \times 100$$

$$= \left|\frac{0.06238 - 0.06242}{0.06238}\right| \times 100$$

$$= 0.0716\%$$



Iteration 3

The estimate of the root is

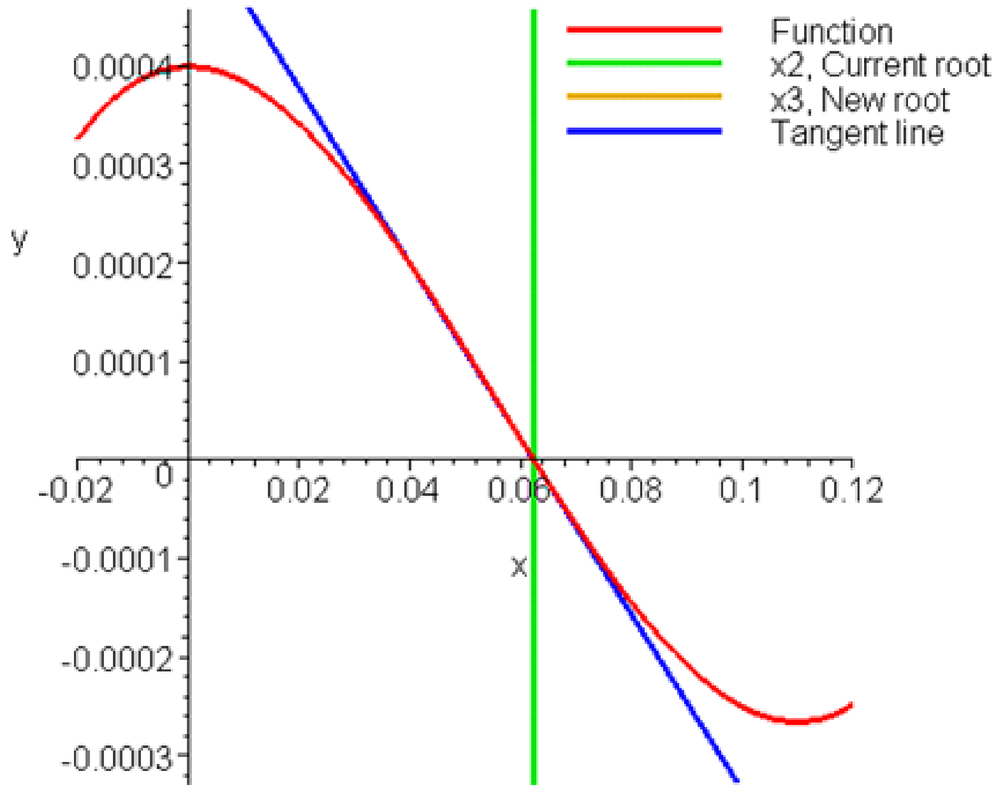
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.06238 - \frac{(0.06238)^3 - 0.165(0.06238)^2 + 3.993\times10^{-4}}{3(0.06238)^2 - 0.33(0.06238)}$$

$$= 0.06238 - \frac{4.44\times10^{-11}}{-8.91171\times10^{-3}} = 0.06238 - \left(-4.9822\times10^{-9}\right) = 0.06238$$

$$|\in_a| = \left|\frac{x_2 - x_1}{x_2}\right|\times100$$

$$= \left|\frac{0.06238 - 0.06238}{0.06238}\right|\times100$$

$$= 0\%$$



Advantages

- Converges fast (quadratic convergence), if it converges.

- Requires only one guess

Drawbacks

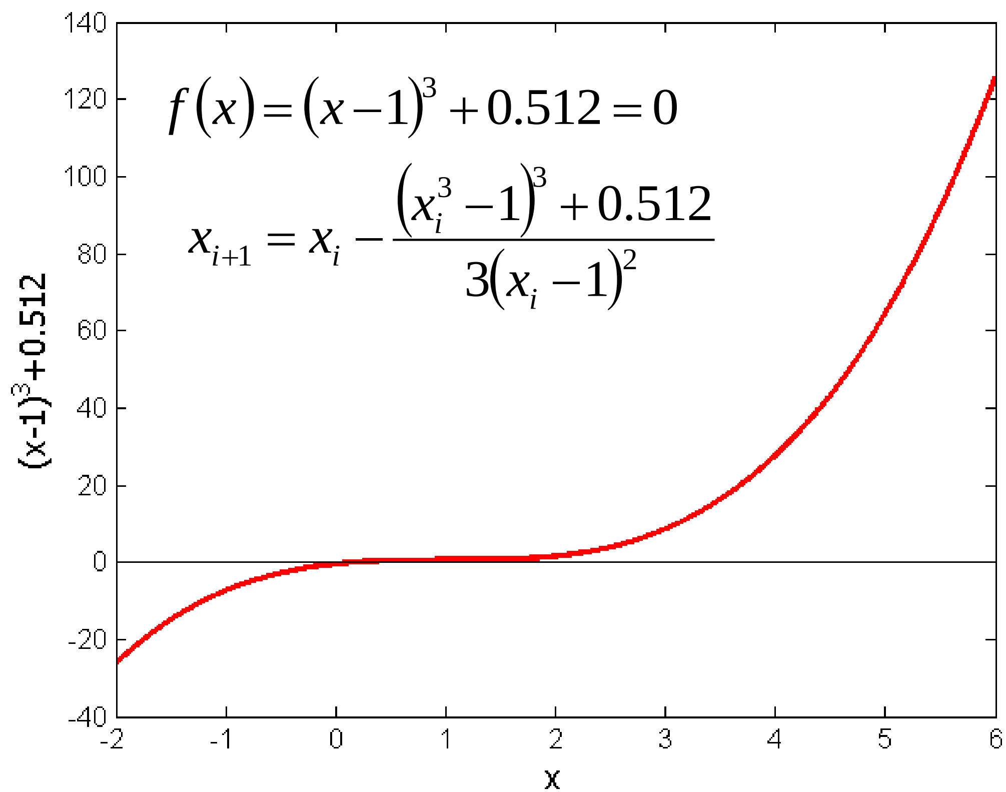
1. Divergence at inflection points

Selection of the initial guess or an iteration value of the root that is close to the inflection point of the function f(x) may start diverging away from the root in the Newton-Raphson method.

The root starts to diverge at Iteration 6 because the previous estimate of 0.92589 is close to the inflection point of $x = 1$.

Eventually after 12 more iterations the root converges to the exact value of $x = 0.2$.

Iteration Number	x_i
0	5.0000
1	3.6560
2	2.7465
3	2.1084
4	1.6000
5	0.92589
6	−30.119
7	−19.746
18	0.2000

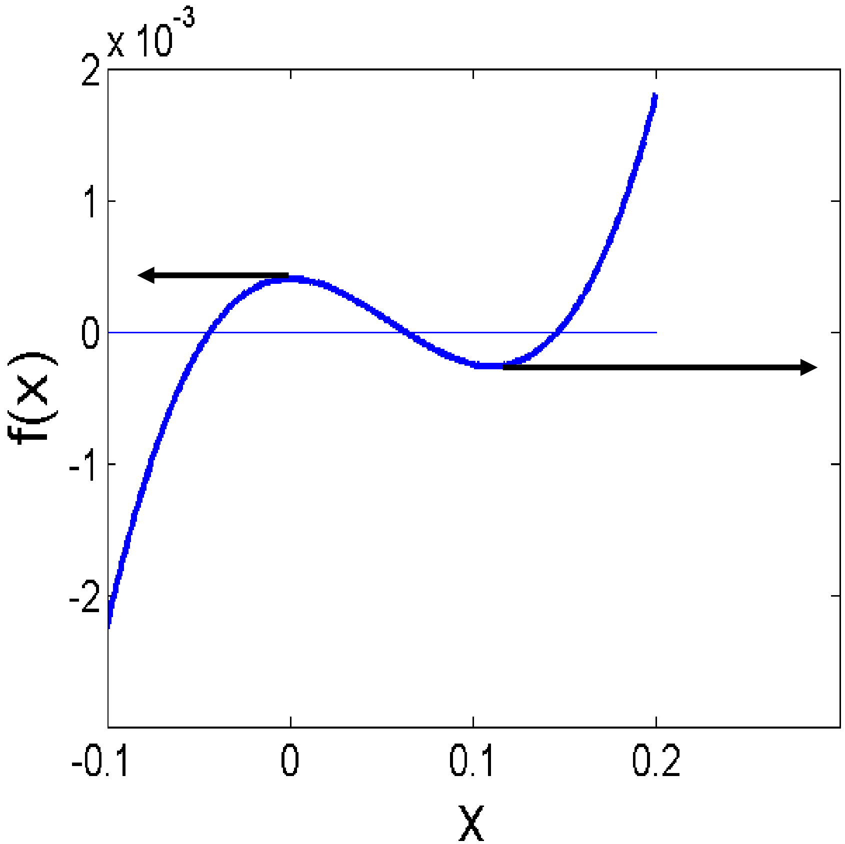


Drawbacks

2. Division by zero

For the equation $f(x) = x^3 - 0.03x^2 + 2.4\times10^{-6} = 0$

$$x_{i+1} = x_i - \frac{x_i^3 - 0.03x_i^2 + 2.4\times10^{-6}}{3x_i^2 - 0.06x_i}$$



For $x_0 = 0$ or $x_0 = 0.02$**, the denominator will equal zero.**

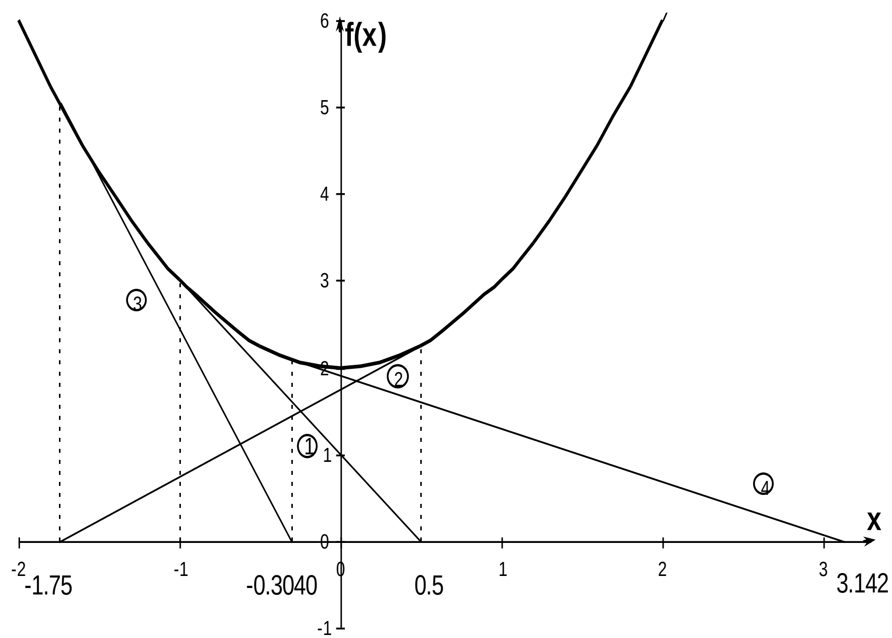
Drawbacks

3. Oscillations near local maximum and minimum

Results obtained from the Newton-Raphson method may oscillate about the local maximum or minimum without converging on a root but converging on the local maximum or minimum.
Eventually, it may lead to division by a number close to zero and may diverge.

For example for $f(x) = x^2 + 2 = 0$ **the equation has no real roots.**

Iteration Number	x_i	$f(x_i)$	$\lvert\in_a\rvert\%$
0	–1.0000	3.00	
1	0.5	2.25	300.00
2	–1.75	5.063	128.571
3	–0.30357	2.092	476.47
4	3.1423	11.874	109.66
5	1.2529	3.570	150.80
6	–0.17166	2.029	829.88
7	5.7395	34.942	102.99
8	2.6955	9.266	112.93
9	0.97678	2.954	175.96



Drawbacks

4. Root Jumping

In some cases where the function f(x) is oscillating and has a number of roots, one may choose an initial guess close to a root. However, the guesses may jump and converge to some other root.

For example

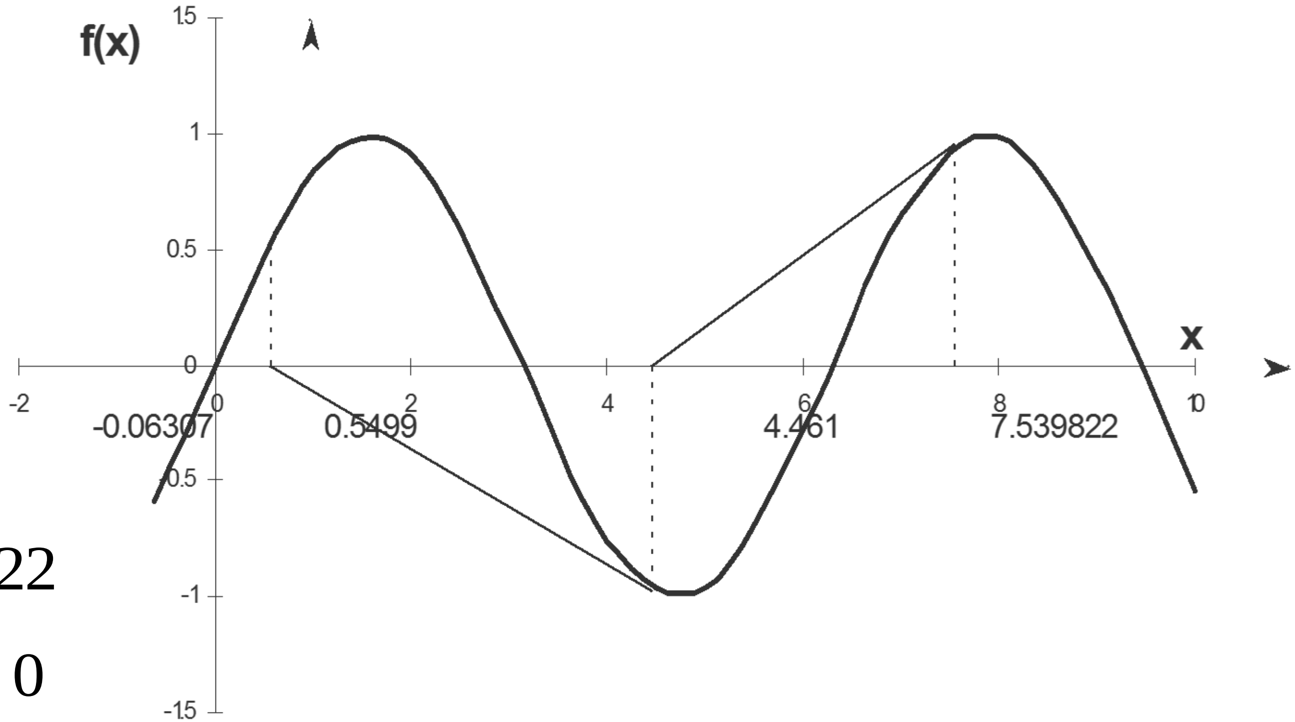
$$f\left(x\right) = \sin x = 0$$

Choose

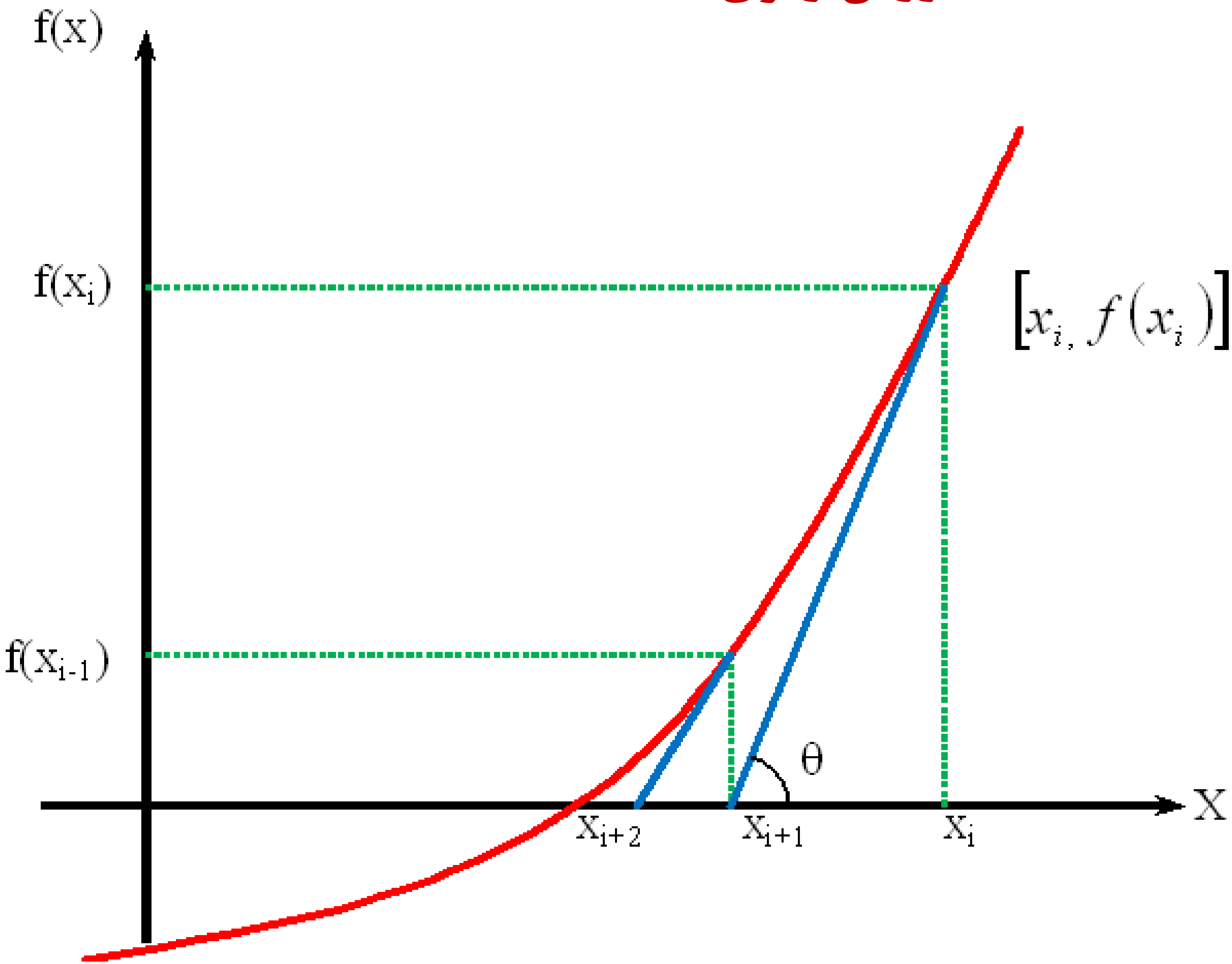
$$x_0 = 2.4\pi = 7.539822$$

It will converge to $x = 0$

instead of $x = 2\pi = 6.2831853$



روش وتری(Secant Method)



Geometrical illustration of the Newton-Raphson method.

Newton's Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \textbf{(1)}$$

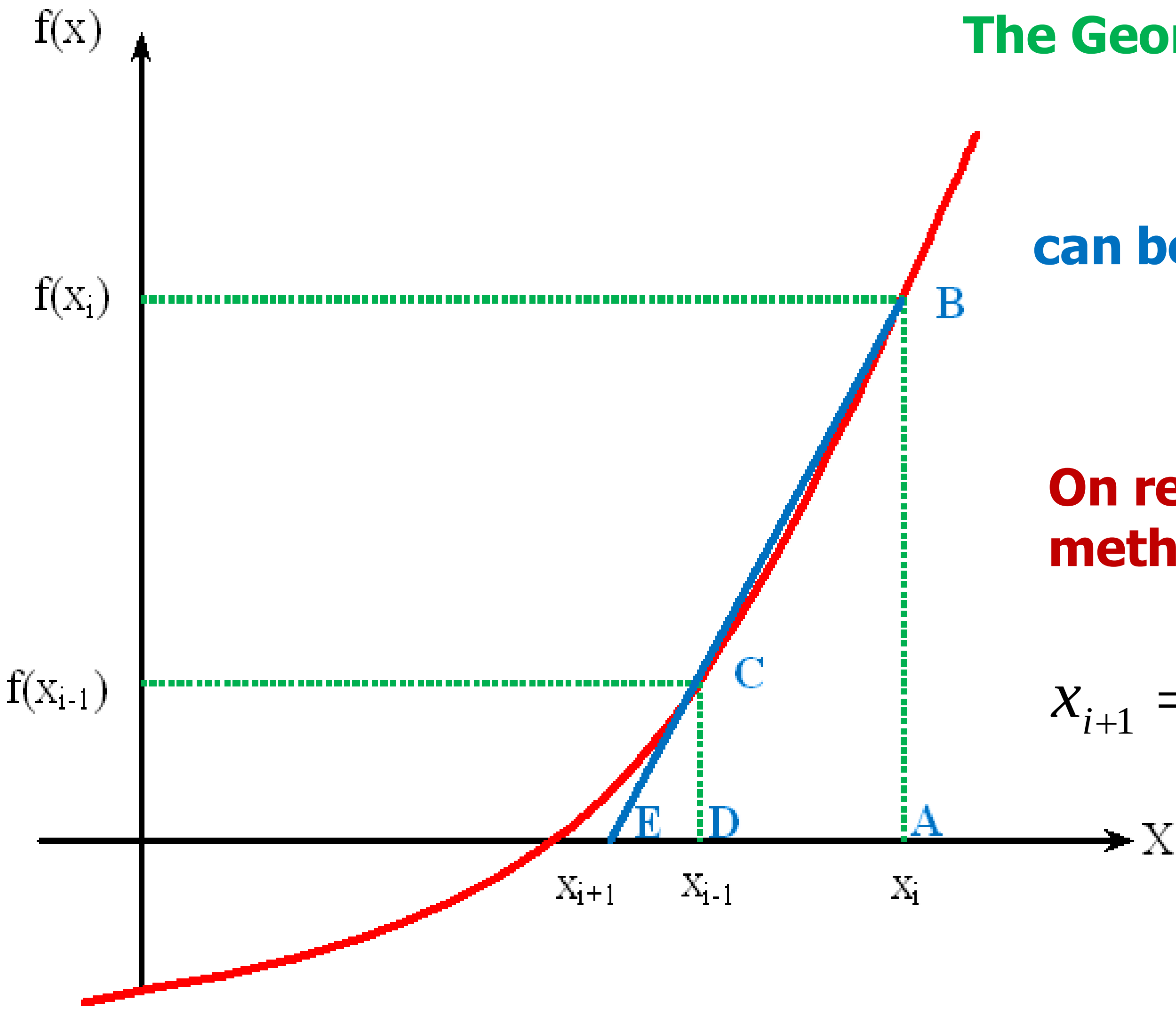
Approximate the derivative

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \quad \textbf{(2)}$$

Substituting Equation (2) into Equation (1) gives the Secant method

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

The secant method can also be derived from geometry:



The Geometric Similar Triangles

$$\frac{AB}{AE} = \frac{DC}{DE}$$

can be written as

$$\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

On rearranging, the secant method is given as

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Algorithm for Secant Method

Step 1: Calculate the next estimate of the root from two initial guesses

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Find the absolute relative approximate error $|\in_a| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100$

Step 2:

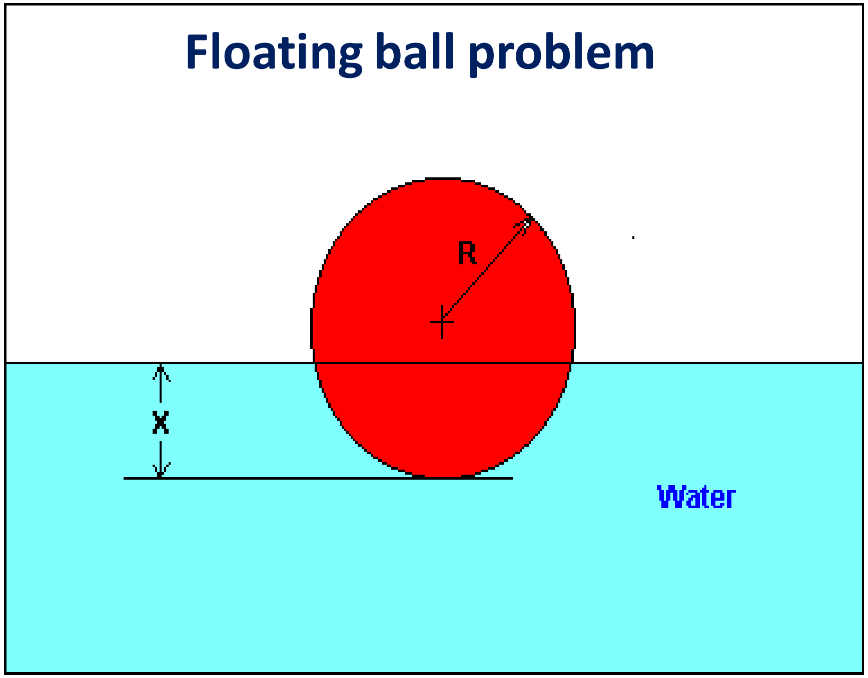
Find if the absolute relative approximate error is greater than the prespecified relative error tolerance.

If so, go back to step 1, else stop the algorithm.

Also check if the number of iterations has exceeded the maximum number of iterations.

Example

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$



Use the Secant method of finding roots of equations to find the depth *x* to which the ball is submerged under water.

- **Conduct three iterations to estimate the root of the above equation.**
- **Find the absolute relative approximate error at the end of each iteration.**

Let us assume the initial guesses of the root of $f(x)=0$ as $x_{-1}=0.02$ and $x_0=0.05.$

Iteration 1

The estimate of the root is

$$x_1 = x_0 - \frac{f(x_0)(x_0 - x_{-1})}{f(x_0) - f(x_{-1})}$$

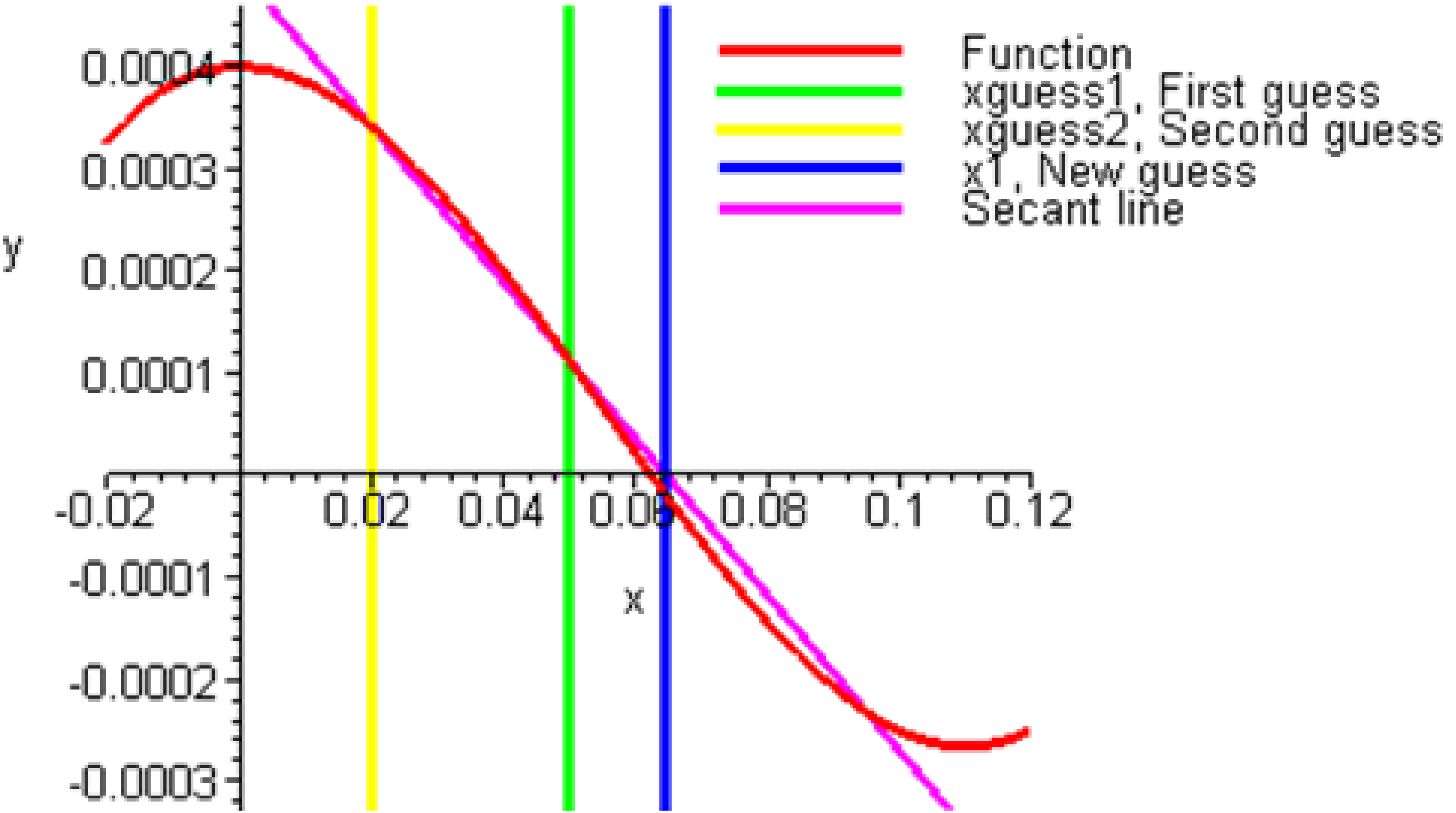
$$= 0.05 - \frac{\left(0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right)(0.05 - 0.02)}{\left(0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right) - \left(0.02^3 - 0.165(0.02)^2 + 3.993\times10^{-4}\right)}$$

$$= 0.06461$$

$$|\epsilon_a| = \left|\frac{x_1 - x_0}{x_1}\right| \times 100 = \left|\frac{0.06461 - 0.05}{0.06461}\right| \times 100 = 22.62\%$$

you need an absolute relative approximate error of 5% or less.

Graph of results of Iteration 1



Iteration 2

The estimate of the root is

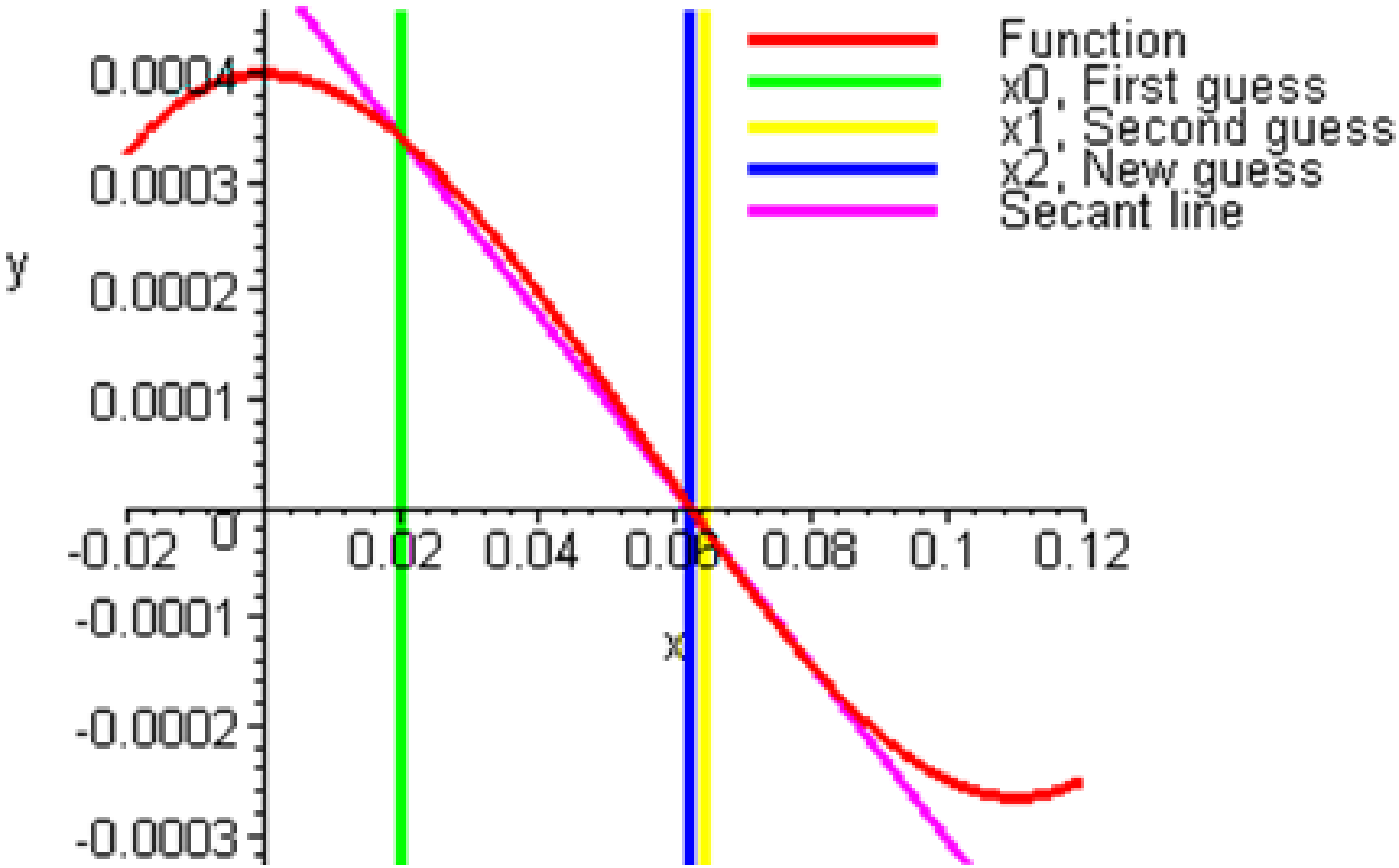
$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= 0.06461 - \frac{\left(0.06461^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right)(0.06461 - 0.05)}{\left(0.06461^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right) - \left(0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}\right)}$$

$$= 0.06241$$

$$|\epsilon_a| = \left|\frac{x_2 - x_1}{x_2}\right| \times 100 = \left|\frac{0.06241 - 0.06461}{0.06241}\right| \times 100 = 3.525\%$$

Graph of results of Iteration 2



Iteration 3

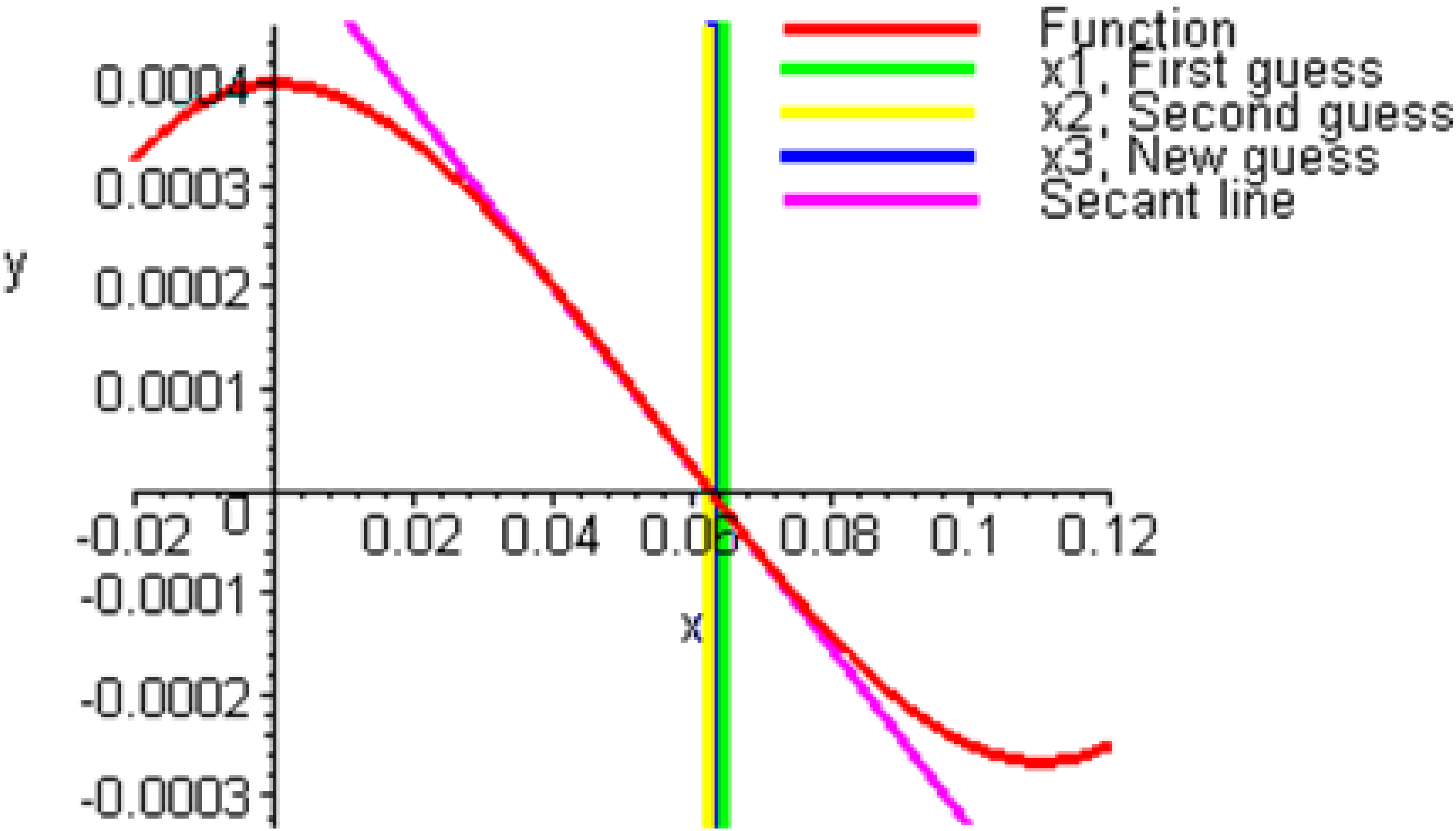
The estimate of the root is

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$$= 0.06241 - \frac{\left(0.06241^3 - 0.165(0.06241)^2 + 3.993\times10^{-4}\right)(0.06241 - 0.06461)}{\left(0.06241^3 - 0.165(0.06241)^2 + 3.993\times10^{-4}\right) - \left(0.05^3 - 0.165(0.06461)^2 + 3.993\times10^{-4}\right)}$$

$$= 0.06238$$

$$\left|\in_a\right| = \left|\frac{x_3 - x_2}{x_3}\right|\times100 = \left|\frac{0.06238 - 0.06241}{0.06238}\right|\times100 = 0.0595\%$$

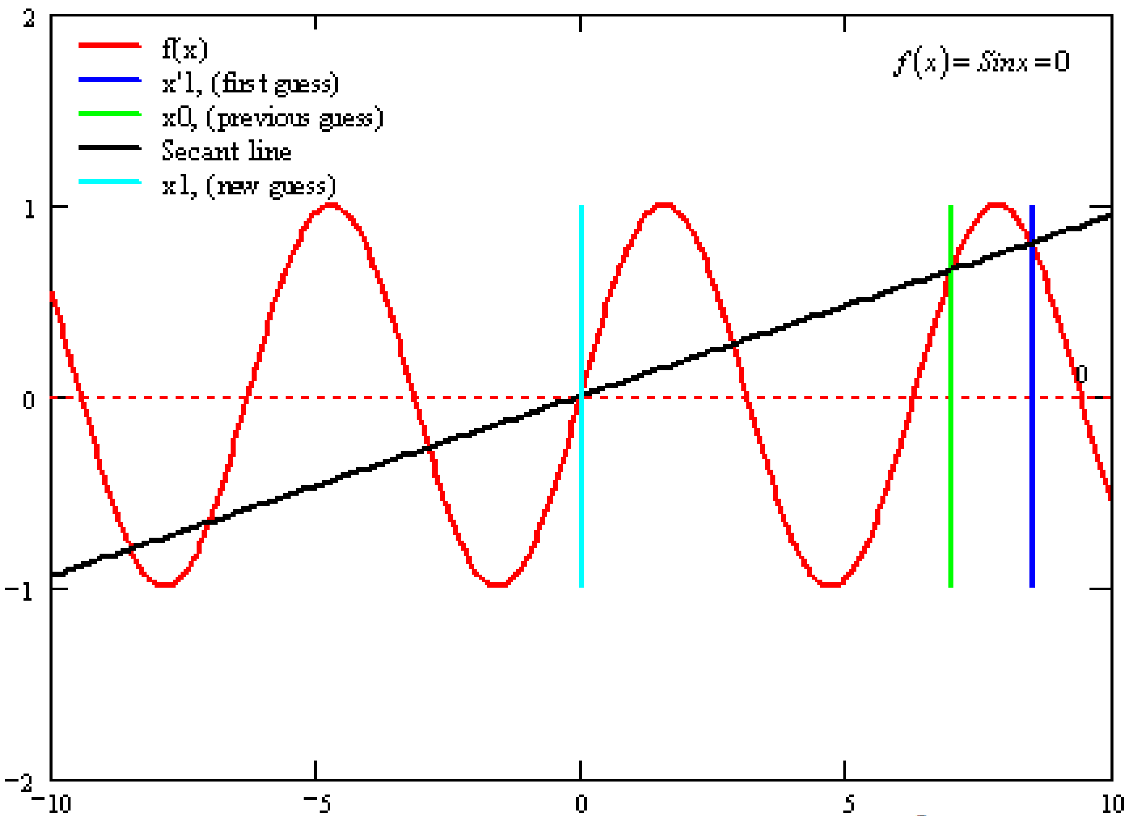

Function
x1, First guess
x2, Second guess
x3, New guess
Secant line
y
0.0004
0.0003
0.0002
0.0001
0
-0.0001
-0.0002
-0.0003
-0.02
0.02
0.04
0.06
0.08
0.1
0.12
x

Advantages

- **Converges fast, if it converges**
- **Requires two guesses that do not need to bracket the root**

Drawbacks

- **Division by zero**
- **Root Jumping**



روش تکرار ساده

در این روش پس از آزمون شرایط موجود بودن ریشه برای معادله $f(x)=0$ در فاصله [a b]،
معادله $f(x)=0$ را پس از دستکاریهایی به صورت $x=g(x)$ نوشته می شود.

به طوری که α ریشه هر دو معادله باشد، یعنی:

$$f(\alpha)=0 \quad \& \quad \alpha=g(\alpha)$$

نکته۱: معمولا از روی معادله $f(x)=0$ به صورتهای مختلفی می توان به شکل $x=g(x)$ رسید.

$$f(x) = x^2 - x - 2 \quad \begin{cases} g(x) = x^2 - 2 \\ g(x) = \sqrt{x+2} \\ g(x) = 1 + \dfrac{2}{x} \end{cases}$$

نکته ۲. بدیهی ترین شکل ممکن برای تبدیل معادله $f(x)=0$ به صورت $x=g(x)$ عبارتند از:

$$x = x - f(x)$$

$$x = x + f(x)$$

پس از نوشتن معادله $f(x)=0$ به صورت $x=g(x)$، هرگاه x_0 تقریبی از α، ریشه معادله باشد x_nها به طریق زیر ساخته می شوند.

$$x_1 = g(x_0)$$

$$x_2 = g(x_1)$$

$$x_3 = g(x_2)$$

$$\vdots$$

$$x_{n+1} = g(x_n), \qquad n = 0, 1, 2, \ldots$$

شرط کافی برای همگرایی روش تکرار ساده

- برای $x \in [a\ b]$ داشته باشیم $g(x) \in [a\ b]$
- برای $x \in [a\ b]$ داشته باشیم $|g'(x)| < 1$

نکته: شرایط فوق شرایط کافی هستند، بنابراین هرگاه تابع g(x) دارای دو شرایط فوق باشد x_nها به α میل می کند.

اگر g(x) حداقل یکی از شرایط را نداشته باشد، آنگاه g دیگری را انتخاب می کنیم.

مثال. برای تعیین تقریب ریشه معادله $f(x) = 3xe^x - 1 = 0$ که در فاصله (0 1) قرار دارد از روش تکرار ساده استفاده نمایید.

با فرض $x_0 = 0.5$ و تقریب 3D مسئله را حل کنید.

$$g(x) = \frac{e^{-x}}{3}$$

$x \in (0,1)$ ⇨

$$0 < x < 1$$

$$-1 < -x < 0$$

$$e^{-1} < e^{-x} < e^{0}$$

$$\frac{1}{3e} < \frac{e^{-x}}{3} < \frac{1}{3}$$

⇨

$$\boxed{0 < 0.12 = \frac{1}{3e} < g(x) < \frac{1}{3} < 1}$$

لذا برای $x \in (0,1)$ داریم: $g(x) \in (0,1)$

$$g'(x) = \frac{-e^{-x}}{3} \xrightarrow{x \in (0,1)} |g'(x)| = \frac{e^{-x}}{3} < \frac{e^{0}}{3} = \frac{1}{3} < 1$$

بنابراین g(x) به طور مناسب انتخاب شده است.

$x_0 = 0.5$

$$x_{n+1} = \frac{e^{-x_n}}{3}$$

$x_1 = 0.2022$

$x_2 = 0.2723$

$x_3 = 0.2539$

$x_4 = 0.2586$

$x_5 = 0.2574$

$x_6 = 0.2577$

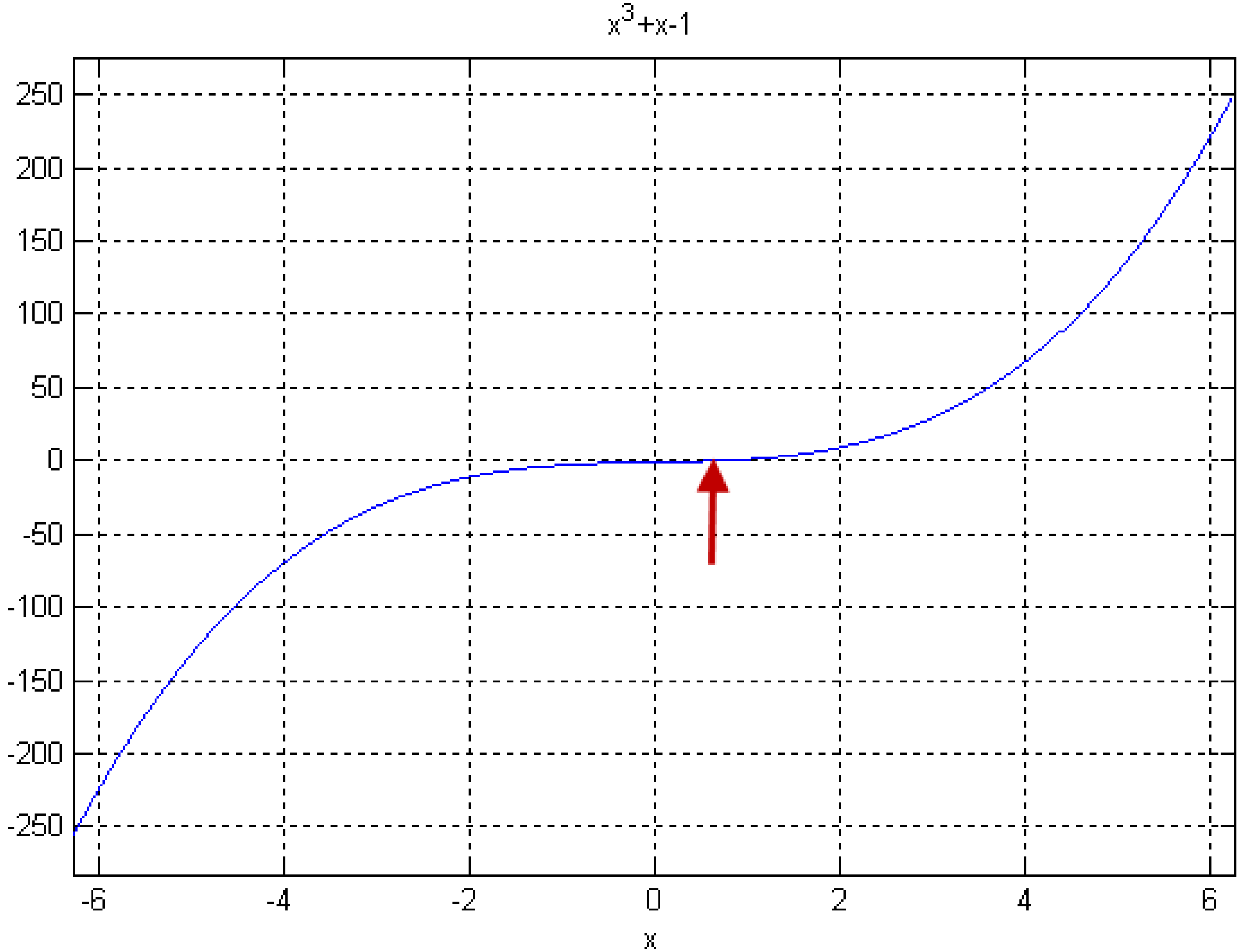
$x_7 = 0.2576 \Rightarrow x_8 = 0.2576$

$\alpha \cong 0.258 \quad (3D)$

```
>> f1 = @(x)x^3+x-1;

>> ezplot(f1)

>> grid on
```




```
>> r=fzero(f1,0)

r =

    0.6823
```